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INTRODUCTION

This report describes the work performed on the Digital Adaptive Control Research project during the fourth quarterly period ending 31 January 1964. In order to provide a consistant level of documentation, this report covers only work performed during the fourth quarter. Summary report No. 1544-5 is a synoptic survey of the complete study.

Section II comprises the Theoretical Studies made during this period, and includes a general basis for controls of the type being studied based on plant description by Volterra Series. The previously developed control methods of the project are identified as special cases.

Section III identifies the Computational Requirements of a control computer for this system. Feasibility of on-line operation with relatively modest computers is shown.

Section IV presents a variety of Experimental Results obtained during the period. Most notable is the hybrid simulation with filtering, which is a simulation of the complete control process.

Section V is a glossary of all equation and simulation symbols used in this report and all previous progress reports.

Section II is largely abstracted from a paper "Control Without Model or Plant Identification" by J. Zaborszky and W. Humphrey, authored during the period, and submitted to the papers committee of the 1964 JACC. Some non-substantive revisions have been made by J. Zaborszky and E. Buder in preparing this report. Section III is the work of R. Janitch, and is based on his previous work in programming the EM-5000 computer in the hybrid simulator. Section III and IV were compiled by R. Janitch and L. Woltmann based on hybrid simulation runs made almost continuously in the reporting period. Section V was compiled by E. Buder.

The analytical studies of Section II were performed under contract NASw-599. The simulation studies (Section III and IV) were funded under Emerson Electric's R & D program.

AUTHOR

II. THEORETICAL STUDIES

Introduction

A pilot flying an airplane is constantly identifying the plane's behavior without doing so in terms of coefficients of differential equations, transfer function, or even in terms of Volterra kernels. What the pilot senses is simply the current response of the plane alongside its current sensitivity to control action, both on the basis of the immediate past and extrapolated into the near future.

Because the human pilot is an excellent adaptive system, within his limitations of speed, the success of adaptive control may lie along lines characterizing his operation. This paper attempts to explore such an approach.

Assumptions regarding the controlled plant in this paper are very general and valid for almost all physical equipment. Specifics like assumptions of linearity or a particular order or of slow variation of the system are avoided in the general development. Information regarding the behavior of the plant is to be derived solely from potentially noisy measurements. The output quantity and the control variable, including possibly a few derivatives, are available for measurement; but not the complete "state" vector, the dimensionality of which is unknown under the assumptions made.

Assumptions

The application of the control method of this paper is restricted to systems which produce continuous and bounded outputs, $x(t)$, when excited by continuous and bounded inputs, $u(t)$. The input-output relationship of such a system is a functional which maps the Banach space of continuous functions over an interval onto itself. If such a functional is continuous it can be approximated over finite time intervals arbitrarily closely by a finite functional polynomial of the form;

$$1. \quad x(t) = y(t) + \sum_{j=1}^{J} \int_0^t \cdots \int_0^t h_j(t\ \tau_1 \cdots \tau_j)\, u(\tau_1) \cdots u(\tau_j)\, d\tau_1 \cdots d\tau_j$$

where h_j are the kernels of the functional polynomial fit.

If the functional is analytic it can be represented by an infinite series ($J = \infty$) of the type of equation 1, a so called Volterra Series or functional Taylor Series; h_j are then the Volterra kernels. In Equation 1, $y(t)$ represents the free response that would occur in the absence of any control input, $u(t)$. It must be remembered, however, that equation 1 does not imply superposition. The reason is that the h_j kernels are not unique, they depend on $y(t)$ just like the coefficients of an ordinary Taylor Series depend on the point around which the expansion is obtained.

Note that only the existence of a relation of the form of Equation 1 is assumed, not a knowledge of the kernels or any intention to identify them. This takes in a very broad class of systems. Continuous nonlinearities and time variations are permitted without assuming any particular order of the differential equations or any knowledge of the speed of variation or the existence of the nonlinearity. About the only features excluded are discontinuous nonlinearities such as relays in the plant, but of course, if there are any relays in a control system, they are not likely to be in the plant. Discontinuous time variations are permissible if their occurrences are well recognizable such as the staging of a missile.

Extensions to more than one output or control variable are direct.

Representation of the Response of the Plant and Its Sensitivity to Control Action

The specific control variable functions considered in this study are piecewise constant.

$$2. \quad u(t) = u_k \qquad kT \leq t < (k+1)\,T \qquad |u_k| \leq U$$

This form of the control variable is almost inherent in any control which relies on an in-line digital computer as is assumed here.

The present time will be $t = nT$ and an nT second length section of the latest signals $x(t)$ and $u(t)$ will be kept in the computer memory. Then for $t \geq 0$ using a functional power series of the type of Equation 1 for the interval $0 < t < nT$ and substitutuing Equation 2;

$$3. \quad x(t) = y(t) + \sum_{j=1}^{J} \sum_{k_1=0}^{n-1} \cdots \sum_{k_j=0}^{n-1} A_{k_1 \ldots k_j}(t)\, u_{k_1} \ldots u_{k_j}$$

where by Equation 1 through 3;

$$4. \quad A_{k_1 \cdots k_j}(t) = \begin{cases} \int_{k_1 T}^{K} \cdots \int_{k_j T}^{K} h_j(t, \tau_1 \ldots \tau_j)\, d\tau_1 \ldots d\tau_j & \chi T \leq t \leq (\chi+1)T \\ \int_{k_1 T}^{(k+1)T} \cdots \int_{k_j T}^{(k_j+1)T} h_j(t, \tau_1 \ldots \tau_j)\, d\tau_1 \ldots d\tau_j & (\chi+1)T \leq t \end{cases}$$

with $\chi = \max(k_1, k_2, \cdots k_j)$

$$K = \begin{cases} (k_i+1)T & \text{for } k_i < \chi \\ t & \text{for } k_i = \chi \end{cases}$$

Equation 3 can be rearranged as;

$$5. \quad x(t) = y(t) + \sum_{k=0}^{n-1} \sum_{r=1}^{R} A_{k^r}(t) \, u_{k^r}$$

where usually R = J and $a_{k^r}(t)$ stands for $a_{k,k,\ldots k}$ with k repeated r times and from Equations 3 and 4, considering the symmetry of the kernels;

$$6. \quad a_{k^r}(t) = A_{k^r}(t) + \sum_{i=1}^{k} (r+1) A_{k^r,\,k-i}(t) \, u_{k-i} + \sum_{i=1}^{k} \sum_{j=1}^{i} \frac{4-3\delta_{ij}}{2} \cdot \frac{(r+2)!}{r!} A_{k^r,k-i,k-j}(t) \, u_{k-i} \, u_{k-j} + \ldots$$

where δ_{ij} is the Kronecker Delta and with reference to Equation 4 $A_{k^r,k-i,k-j}$ stands for $A_{k,k,\ldots k,k-i,k-j}$ with k repeated r times.

or more generally,

$$6a. \quad a_{k^r}(t) = \frac{1}{r_1 \, u_k} \sum_{\langle k \rangle} A_{\langle k \rangle} \, M_{\langle k \rangle} \, U_{\langle k \rangle}$$

where

$$6b. \quad M_{\langle k \rangle} = \sum_{r_2=0}^{R-r_1} \sum_{r_3=0}^{R-r_1-r_2} \cdots \sum_{r_R=0}^{R-r_1 \ldots -r_R} \frac{\left(\sum_{i=1}^{R} r_i\right)!}{\prod_{i=1}^{R} r_i!}$$

$$6c. \quad U_{\langle k \rangle} \quad u_k^{r_1} \, u_{k-1}^{r_2} \cdots u_{k-R}^{r_R}$$

$$6d. \quad A_{\langle k \rangle} = A_{k^{r_1}(k-1)^{r_2} \ldots (k-R)^{r_R}}(t)$$

and $\sum_{\langle k \rangle}$ denotes sumation over all different $A_{\langle k \rangle}$ which have significant contributions.

Defining vectors:

$$7. \quad \underline{x} = x^{(i)}\big]_{Nx1} \qquad \underline{y} = y^{(i)}\big]_{Nx1} \qquad \underline{u_k} = u_k{}^r\big]_{Nx1}$$

$$\underline{A}_k(t) = a_{k^r}^{(i)}\big]_{Rx1} \qquad i = 0, 1, \ldots N-1 \quad r = 1, 2, \ldots R$$

where $\underline{x}$ can be a state vector if N is the order of the system from Equation 5

$$8. \quad \underline{x}(t) = \underline{y}(t) + \sum_{k=0}^{n} \underline{A}_k(t)\,\underline{u}_k \qquad t \geq 0$$

Such a representation is possible for the class of plants considered since for this class, functions y(t) and $A_{\langle k\rangle}(t)$ (where $\langle k\rangle$ indicates any of the ordered sets of subscripts used in equation 6) will be continuous and repeatedly differentiable with respect to t, except possibly at t = iT, for i an integer, where higher derivatives of $A_{\langle k\rangle}(t)$ will be discontinuous. Then truncated Taylor series representations can be found for, respectively, y(t) and $A_{\langle k\rangle}(t)$

$$9. \quad y^{(i)}(t) = \sum_{p=i}^{P} y_p \frac{p!}{(p-i)!} t^{p-i}$$

$$10. \quad A^{(i)}_{\langle k\rangle}(t) = \begin{cases} 0 & t < kT \\ \sum_{p=i}^{P} A_{\langle k\rangle\, p-} \frac{p!}{(p-i)!} (t-kT)^{p-i} & kT \leq t < (k+1)T \\ \sum_{p=i}^{P} A_{\langle k\rangle\, p+} \frac{p!}{(p-i)!} (t-kT)^{p-i} & (k+1)T \leq t \end{cases}$$

Finally for a continuously, if arbitrarily, time varying plant

$$11. \quad A_{\langle k\rangle\, p\pm} = A_{\langle n-h\rangle\, p\pm} = \sum_{s=0}^{S} A_{\langle n\rangle\, ps\pm} (-hT)^s$$

provided

$$11a. \quad \langle k-h\rangle = \langle k_1-h,\ k_2-h,\ \ldots\ k_j-h\rangle$$

$$\text{if} \quad \langle k\rangle = \langle k_1,\ k_2,\ \ldots\ k_j\rangle \text{ and } k_i - h \geq 0$$

Probably S = 1 is sufficient for most plants.

Equation 8 can be rewritten for $t \geq nT$

$$12. \quad \underline{x}(t) = \underline{x}_n(t) + \underline{A}_n(t)\,\underline{u}_n$$

where

$$13. \quad \underline{x}_n(t) = \underline{y}(t) + \sum_{k=0}^{n-1} \underline{A}_k(t)\,\underline{u}_k$$

represents the current response of the system at $t \geq (nT)$ resulting from its initial state at $t = 0$ (the term $\underline{y}(t)$) and the u_k, $k = 0, \ldots n-1$ control steps applied during $0 \leq t < nT$. The last term in equation 2 identifies the effect of the control variable u_n which will be applied $(n-1)\ T < t < nT$; $\underline{A}_n$ then is the current sensitivity of the system to this force. Of course $\underline{A}_n u_n$ is a column of polynomials in u_n. Note that in spite of its form equation 12 does not represent superposition since the sensitivity $\underline{A}_n(t)$ is not a unique constant of the plant, but a function of past states $y(t)$ and past control forces u_k applied to the plant. Equations 12 and 13 represent a kind of "canonical equations" describing the current expected behavior of a plant which may be linear or nonlinear, stationary or time varying. These canonical or standard equations define the current behavior of this general class of plants when controlled digitally with a first order hold. The coefficients of this canonical equation can be computed, if the plant is known, or determined from the signals of the immediate past as is proposed here.

Both current response $x_n(t)$ and current sensitivity $\underline{A}_n(t)$ are fully determined by equation 6-12 provided the present parameters $A_{\langle n \rangle ps\pm}$ and y_p are identified. This represents identification of the current response and sensitivity to the next input step. It does not, however, identify the plant in the normal sense.

A plant is identified when a relationship (differential equation, transfer function, Voltera series, etc.) is established for it which permits computation of the plant output for an arbitrary input and an arbitrary initial state. What is identified in this study permits only the prediction of the response for the existing past conditions of state and control forces applied in the past, under the influence of the control step ahead which is of a strongly limited nature. In this sense then it is not plant identification, but identification of current response and current sensitivity to control force.

Determining the Coefficients

Considering equations 9 and 10, equation 8 can be rewritten in the form of

$$14. \quad \underline{x}(t) = \sum_{e=0}^{P} g_{ke} t^e \qquad kT \leq t < (k+1)T$$

which is simply a Taylor series expansion (ideally $P = \infty$) of the output and its derivatives. The coefficients g_{ke} are, by equations 9 through 11, linear combinations of the $A_{\langle n \rangle ps\pm}$ and the y_p coefficients. A different combination will arise for every interval unless $u_k = u_i$ for all k and i. Consequently there will be a separate series of the form of 13 for every interval T.

Now if it is assumed that the signal $\underline{x}(t)$ can be measured exactly without noise effect then a definite set of g_{ke} can be established for each interval $kT \leq t < (k+1)T$. Equating these to the expressions for g_{ke}

obtained from equations 6 through 11 a set of simultaneous linear equations results which uniquely determine the $A_{\langle n\rangle ps\pm}$ and y_p coefficients provided the number of coefficients and intervals is properly coordinated. Specifically for the first term in g_{ke}

$$15.\quad g_{ke} = \sum_{s=o}^{S} \sum_{p=e}^{P} \binom{p}{e} \sum_{\langle n\rangle} M_{\langle n\rangle} \left[\sum_{h=n-k-1}^{n} U_{\langle n-h\rangle} A_{\langle n\rangle ps+} (-hT)^s (-(n-h)T)^{p-e} \right.$$

$$\left. + U_{\langle n-h\rangle} A_{\langle n\rangle ps-} (-(n-k+1)T)^s ((k-1)T)^{p-e} \right] + y_e$$

$$\begin{matrix} k = 0, 1, 2, \ldots n \\ e = 0, 1, 2, \ldots p \end{matrix}$$

where notations $\sum_{\langle n\rangle}$, $M_{\langle n\rangle}$, and $U_{\langle n-h\rangle}$ are defined in connection with equation 6.

This will yield a sufficient number of equations provided

$$16.\quad n = 2\mathcal{M}(S+1)+1$$

where $\mathcal{M}$ is the number of the $\langle n\rangle$ sets which are considered significant and the determination of which is desired.

Equations 15 will be independent provided the u_{ki} are not all identical as they would be, for instance, when the limit U of $|u|$ is called for continually. When this latter situation arises, it still would be possible to determine combination coefficients ($g_{ke} = g_e$) which will predict the response as long as $u = U$ is maintained, but any evaluation of the sensitivity to the choice of u would be lost. Basically a different control policy from the one considered in this paper is called for when the available control force is so limited that $|u| = U$ is used most of the time. Although an assumption of exact noisefree measurement of $\underline{x}(t)$ is not realistic, it is no less realistic than assuming a perfectly identified plant and a perfectly identified state vector which are the basis of the major part of the extensive optimal control literature. In both cases these idealized assumptions have value in the sense of establishing idealized reference points.

When the measurements of $\underline{x}(t)$ and possibly also of $\underline{u}_k$ are noisy, the problem of determining the $A_{\langle n\rangle ps\pm}$ and y_p coefficients changes from the direct exact computation of equation 15 into the correstponding statistical problem of the optimum estimation of a set of parameters on the basis of measurements. A wealth of statistical techniques has been developed for this general problem, the choice depending on the nature of the uncertainties in the measurements and the extent and character of statistical information available. This problem will not be pursued

further in this paper, although work is continuing for establishing guidelines of choice in this area. Before passing on, a mean square estimation for stationary noise will be outlined, nevertheless, as an example (not implying the preferred technique).

Let us assume that measured values $\underline{\xi}(t)$ and ν_k respectively of $\underline{x}(t)$ and of u_k are contaminated by stationary, uncorrelated noise.

17. $$\underline{\xi}(t) = \underline{x}(t) + \underline{n}(t)$$

18. $$\nu_k = u_k + m_k$$

and that the noise swamps all coordinates of $\underline{x}$ above the Z-th measured derivative. Then let the following nz integrals be established

19. $$I\left\{k_i k_j\right\}_z = \int_{k_i T}^{k_j T} \left[\xi_z(t) - \left(\sum_{e=z}^{P} \frac{e!}{(e-z)!} g_{ke}\, t^{e-z}\right)\right]^2 dt \qquad 0 \leq z \leq Z$$

where g_{ke} is as defined in equation 15, but u_k is replaced by ν_k, and $\left\{k_i k_j\right\}$ are n distinct sets $0 < k_i < n$, $0 < k_j < n$, and $k_j > k_i$.

Necessary conditions for optimum mean square estimates of $A_{\langle n \rangle ps\pm}$ and y_e are then

20. $$\sum_{z=0}^{Z} \frac{1}{\lambda_z} \frac{\partial I_{\left\{k_i\, k_j\right\} z}}{\partial A_{\langle n \rangle ps\pm}} = 0 \quad \text{or} \quad \sum_{z=0}^{Z} \frac{1}{\lambda_z} \frac{\partial I_{\left\{k_i k_j\right\} z}}{\partial y_e} = 0$$

where the λ_z are weighting factors such as the variances of the corresponding noise.

Equation 20 yields a set of linear equations equivalent to the set of equations 15 in the noiseless case. The integrals implied by equation 20 would be found numerically from the measured data.

Control Policy

Equations 12 and 13 can be rewritten for $t = (n+1)T$

21. $$\underline{x}((n+1)T) = \underline{x}_n((n+1)T) + \underline{A}_n((n+1)T)\,\underline{u}_n$$

22. $$\underline{x}_n((n+1)T) = \underline{y}((n+1)T) + \sum_{k=0}^{n-1} \underline{A}_k((n+1)T)\,\underline{u}_k$$

At this point it is necessary to specify the desired value of $\underline{x}$ at $t = (n+1)T$. This will be denoted by $\underline{c}\ (\ (n+1)T)$ and is an N dimensional vector, which is independent of u_k. $\underline{x}$ and $\underline{c}$ are not actually state vectors since the order of the plant is assumed to be unkown. Furthermore, all physical equipment has a very high actual order or is, in ultimate analysis, a distributed constant system. Consequently, in all practical control situations there are large numbers of uncontrollable modes and the order of the system used in control considerations is only an estimate of the number of modes which are relatively more important. Fortunately, there is a tendency for satisfactory control for a reasonable estimate of the order used for control considerations. This will be illustrated in the sequel.

In this spirit N will be regarded simply as a fixed number which is not higher than the order of the system.

The aims of the control could be identified by a variety of performance criteria. To make this discussion more concrete the following specific criterion will be used.

$$23.\quad \underset{u_n}{\text{Min}} \left[\underline{x}'\ (\ (n+1)T) - \underline{c}'\ (\ (n+1)T)\right] \underline{K} \left[\underline{x}\ (\ (n+1)T) - \underline{c}\ (\ (n+1)T)\right]$$

where $\underline{K}$ is a positive definite matrix, primes denote transpose of matrices, and

$$24.\quad |u_n| \leq U$$

The aim is then to select u_n in such a way as to minimize the positive definite quadratic form (Euclidian norm if $\underline{K}$ is the unity matrix) in equation 23 subject to the constraint on u_n. Essentially this amounts to reducing to a minimum the distance measured in the N dimensional manifold between the actual and desired states at $t = (n+1)T$. A necessary condition for satisfying the criterion of equation 23 is that the derivative of the bracket with respect to u_n be zero. That is using equation 21.

$$25.\quad \frac{d\underline{u}_n'}{du_n} \underline{A}_n'\ (\ (n+1)T)\underline{K} \left[\underline{x}_n\ (\ (n+1)T) + \underline{A}_n\ (\ (n+1)T)\ \underline{u}_n - \underline{c}((n+1)T)\right] = 0$$

This clearly is an algebraic equation of order (2R-1) in u_n. The order is always odd; consequently, there will always be one real solution which signifies a minimum but the possibility exists for more than one real solution some of which may denote maxima. If the solution for u_n of equation 25 resulting in the smallest minimum exceeds U in absolute value then $u_n = U$ is to be used with the proper sign.

If equation 25 is strongly nonlinear, a complete solution for all roots may be required along with an analysis of which yields the desired minimum. In most cases, however, a simplified way of finding the desired

u_n can be found which is illustrated in the following.

Simplifications

In the broad class of systems considered here, the higher order terms in the above working equations will become progressively smaller as T is reduced. Eventually only the first order terms will be significant. Selecting the optimum T which would be accomplished in a learning process is outside the scope of this paper. It seems, however, that the practical cases would tend to be R = 1 and R = 2, considering that beyond R = 2 the number of terms begins to proliferate prohibitively. It then is desirable to devote some special discussion to the cases of R = 1, 2.

Case R = 1

First the special forms of equations 21, 22, 15, 10, 7 and 16 will be written

26. $\underline{x}((n+1)T) = \underline{x}_n((n+1)T) + \underline{a}_n((n+1)T)u_n$

27. $\underline{x}_n((n+1)T) = \underline{y}((n+1)T) + \sum_{k=0}^{n-1} \underline{a}_k((n+1)T)u_k$

28. $\underline{a}_k = \overset{(i)}{A}_k(t)\Big]$

29. $\overset{(i)}{A}_k(t) = \sum_{p=1}^{P} \sum_{s=0}^{S} A_{nps\pm} \frac{p!}{(p-i)!}(t-kT)^{p-i}(n-k)^s(-T)^s$

30. $g_{ke} = \sum_{p=e}^{P} \sum_{s=0}^{S} \binom{p}{e}\Big[\sum_{h=1-k+n}^{n} u_{n-h} A_{nps+}(-hT)^s(-(n-h)T)^{p-e}$

$+u_{\quad} \cdot A_{\quad\quad} (-(n-k+1))^s((k-1)T)^{p-e}\Big] + v_e$

31. $n = 2(S+1) + 1$

Significantly, an explicit solution of equation 25 becomes possible

32. $u_n = \dfrac{\underline{a}'_n((n+1)T)\underline{K}\left[\underline{x}_n((n+1)T) - \underline{c}((n+1)T)\right]}{\underline{a}'_n((n+1)T)\underline{K}\,\underline{a}_n((n+1)T)}$

If further the parameter variation is slow compared to T, then $\underline{a}_k(t) = \underline{a}$. If reliable values for $\underline{x}$ are available, it is then sufficient to use one interval for finding $\underline{a}$.

33. $\underline{x}(nT) = \underline{x}_{n-1}(nT) + \underline{a}\, u_{n-1}$

Applying truncated Taylor series to estimate $\underline{x}_{n-1}(nT)$

34. $\underline{x}_{n-1}(nT) = \underline{A}\, \underline{x}\, ((n-1)T)$

where

34. $$\underline{A} = \begin{bmatrix} 1 & T & T^2/2 & . & . & T^P/P! \\ 0 & 1 & T & & & T^{P-1}/(P-1)! \\ . & & & & & \\ . & & & & & \\ 0 & 0 & - & - & - & T^{P-N+1}/(P-N+1)! \end{bmatrix} \quad (P+1)xN$$

In which case x((n-1)T) is a P vector and x(nT) is an N vector in equation 34 with $P+1 \geq N$. If $P + 1 = N$, which also represents the actual order of the system and the system happens to be linear, the $\underline{A}$ can be viewed as a crude approximation of the state transition matrix. The actual state transition matrix can be used in these equations for comparison purposes in studies.

Now sensitivity $\underline{a}$ can be computed as

35. $$\underline{a} = \frac{\underline{x}(nT) - \underline{A}\, \underline{x}\, ((n-1)T)}{u_{n-1}}$$

and from Equation 25 relation defining the selection of the control force for the next interval becomes

36. $$u_n = \frac{\underline{a}'\, \underline{K}\, [\underline{A}][\underline{x}(nT) - \underline{c}\, ((n+1)T)]}{\underline{a}'\, \underline{K}\, \underline{a}}$$

$$|u_n| \leq U$$

If the measurements of $\underline{x}$ (nT) are noisy or the plant is fast time varying, then $\underline{a}$ must be estimated from several past intervals. A version of mean square estimation for stationary plants is mentioned here as an example

37. $$\frac{\partial}{\partial a_i} \sum_{k=0}^{K} \left[\underline{x}'((n-k)T) - \underline{x}'((n-k-1)T)\underline{A}' - \underline{a}'u_{n-k-1}\right] \underline{W} \left[\underline{x}((n-k)T) - \underline{A}\underline{x}((n-k-1)T) - \underline{a}u_{n-k-1}\right] = 0$$

here $\underline{W}$ is a weighting matrix and k is the number of past intervals kept in the memory. Equation 37 represents a set of algebraic equiations which is a necessary condition for the best estimate of $\underline{a}$.

Case R = 2

Again the special versions of the working equations for R = 2 will be listed:

38. $$\underline{x}((n+1)T) = \underline{x}_n((n+1)T) + \underline{a}_n((n+1)T)u_n + \underline{a}_n 2((n+1)T)u_n^2$$

39. $$\underline{x}_n((n+1)T) = \underline{y}((n+1)T) + \sum_{k=0}^{n-1}\left[\underline{a}_k((n+1)T)u_k + \underline{a}_k 2((n+1)T)u_k^2\right]$$

40. $$\underline{a}_k = A_k^{(i)}(t) + \sum_{i=0}^{I} A_{k,k-i}^{(i)} u_k \quad \text{and} \quad \underline{a}_k 2 = A_k 2^{(i)}(t)$$

There will then be the following coefficients to find A_{kps-}, A_{kps+}, $A_{k(k-1)ps-}$, $\cdots$ $A_{k(k-I)ps-}$, $A_{k(k-1)ps+}$ $\ldots$ $A_{k(k-I)ps+}$, $A_k^2{}_{ps-}$, $A_k^2{}_{ps+}$, and y_p giving a total of $\eta = 2+I$ in conjunction with equation 16. So if S = 1 which should be usually satisfactory, then it is necessary to keep a total of n = 2(2+I) 2+1 intervals in the memory to identify the response and sensitivity of a system represented by the R = 2 case. The largest possible value of I would, of course, be n but I = 2$\sim$4 will probably be satisfactory giving n = 17$\sim$25. A corresponding number of not completely overlapping intervals would be needed for estimation in conjunction with equation 20. Further reduction of the number of intervals needed (e.g. by choosing S = 0 or I = 0) might be possible in many instances.

Some additional study of equation 25 for the selection of the optimal control force u_n is indicated.

With R = 1, $\underline{A}_n$ reduces to a column matrix $\underline{a}_n$ ((n+1)T) also $\frac{du_n'}{du_n} = 1$, and term $\underline{a}_n u_n$ varies with u_n in a manifold consisting of a straight line which will have a unique nearest point to $\underline{c}$((n+1)T) as given in equation 33 and 36.

With R = 2, $\underline{A}_n$ contains two columns and $\underline{A}_n u_n$ takes the form $\underline{A}_n u_n = \underline{a}_n u + \underline{a}_n 2u^2$ as shown in equation 38 which indicates variation on a parabola in the manifold of the plane containing vectors $\underline{a}_n$ and $\underline{a}_n 2$. Since the distance of $\underline{c}$ from the plane of this parabola is fixed, the

minimum distance between $\underline{c}((n+1)T)$ and $\underline{x}((n+1)T)$ will be reached at the u_n point along the parabola which is nearest to the projection of $\underline{c}$ into the plane of the parabola. It is then easy to see from the geometry of the parabola that if there is only one real root of equation 25, this identifies a minimum. If there are three real roots, one signifies a maximum, and the other two signify minima, the smaller of which is needed. In practical cases where R = 2, the second order effect should be relatively small which geometrically amounts to replacing the straight line of the R = 1 case by a moderately curved line in the vicinity of the desired number. This will result in a limited shift of the optimum u_n value from what results when only the linear term is considered. The other minimum then will be very large at a large u_n. This realization permits the extraction of the pertinent root without resorting to the full solution of the algebraic equation 25. Let, for instance, the third order equation for R = 2 as obtained from equation 25 be:

$$41.\quad 1 + \alpha u + \delta u^2 + \Delta u^3 = 0$$

Where by the assumptions made δ and Δ are small compared to α using second order approximation at first

$$42.\quad 1 + \alpha v_0 + \delta v_o^2 = 0$$

and assuming that v_o can be approximated by

$$43.\quad v_o = u_o + u_1\delta + u_2\delta^2 + \ldots$$

Now substituting equation 43 with equation 42 and assuming zero value for the coefficients of all distinct powers of δ sufficiently assume that equation 43 is a solution of equation 42. This leads to

$$44.\quad u_o = -\frac{1}{\alpha} \qquad u_1 = -\frac{1}{\alpha^3} \qquad u_2 = -\frac{2}{\alpha^5}$$

In general

$$45.\quad u_k = -\left[\frac{1}{\alpha^{2k+1}}\right] q_k \quad \text{where}$$

$$46.\quad q_k = \sum_{i=0}^{k-1} q_i q_{k-i-1} \qquad k > 0 \qquad q_o = 1$$

Now let

$$47.\quad u = v_o + v_1\Delta + v_2\Delta^2 \quad \ldots$$

and substituting this and v_o in equation 41; then equating individual coefficients of Δ^k to zero,

$$v_1 = - \frac{v_o^3}{\alpha + 2 \beta v_o}$$

$$v_2 = - \frac{3 v_o^2 v_1 + \beta v_1^2}{\alpha + 2 \beta v_o}$$

or in general

$$48. \quad v_k = - \frac{\beta \sum_{i=1}^{k-1} v_i v_{k-i} + \sum_{i=0}^{k-1} \sum_{j=0}^{k-i-j} v_i v_j v_{k-i-j-1}}{\alpha + 2 \beta v_o}$$

Equations 44 through 48 then define the desired smallest control leading to the smallest minimum, the one nearest the linear approximation. The last equation converges fast and is readily evaluated by digital computers. This approach avoids a search for the smaller of the two possible relative minima. Extension to higher orders is possible.

Stability

Since the system is assumed to be unknown and unidentified, general conclusions on stability during a control operation cannot be drawn. It can, however, be observed whether the norm of the distance between the state $\underline{x}$ and desired state $\underline{c}$ as measured in the N dimensional manifold is decreasing at least on the average. If so, the particular control operation at least is stable.

The change of the error norm on the basis of equation 21 and 22 in step number (n + 1), assuming for the time being that $\underline{c}\,((n+1)T) = 0$, is

$$E_n = \| \underline{x}\,((n+1)T) \|_{\underline{H}}^2 - \| \underline{x}(nT) \|_{\underline{H}}^2$$

$$49. \quad = \left[\underline{y}'\,((n+1)T) + \sum_{k=0}^{n} \underline{u}_k' \, \underline{A}_k'\,((n+1)T) \right] \underline{H} \left[\underline{y}((n+1)T) + \sum_{k=0}^{n} \underline{A}_k((n+1)T) \underline{u}_k \right]$$

$$- \left[\underline{y}'\,(nT) + \sum_{k=0}^{n-1} \underline{u}_k' \, \underline{A}_k'\,(nT) \right] \underline{H} \left[\underline{y}(nT) + \sum_{k=0}^{n-1} \underline{A}_k(\,n)T) \underline{u}_k \right]$$

where $\underline{u}_k$ for $0 < k < n-1$ are known past values. With $\underline{H} = \underline{I}$, the unit matrix, equation 49 defines the Eucledian norms. Equation 49 can be evaluated during individual runs.

Proving that E_n is negative for all $\underline{x}(nT)$ is sufficient to assure global asymptotic stability by Liapunov's second method and Krassovskii's theorem provided N is the actual order of the system. In fact, stability is obvious under these conditions. Such general conclusion, of value

for reference purposes, might be drawn for specific plants which are known.

Let us then assume that the Volterra series of the plant as per equation 1 is known, and concentrate on the single interval T beginning at t = nT. Then with reference to equation 8,

$$50.\quad \underline{x}\,((n+1)T) = \underline{y}\,((n+1)T) + \underline{A}_n\,((n+1)T)\,\underline{u}_n$$

where $\underline{y}$ is now the free response resulting from state $\underline{x}(nT)$, and with reference to equation 4 and 6,

$$51.\quad \underline{A}_n\,(t) = \left[\begin{matrix}(i)\\ A_{nj}\end{matrix}\right]$$

$$52.\quad \overset{(i)}{A}_{nj} = \frac{d^i}{dt^i_{nT}} \int^{t} \cdots\cdots \int_{nT}^{t} h_j((t), \tau_1, \ldots \tau_j) d\tau_1 \ldots d\tau_j \Bigg|_{t = (n+1)T}$$

So

$$53.\quad E = \left[\underline{y}'\,((n+1)T) + \underline{u}_n'\,\underline{A}_n'\,((n+1)T)\right]\,\underline{H}\left[\underline{y}((n+1)T) + \underline{A}_n\,((n+1)T)\underline{u}_n\right]$$

$$-\underline{x}'\,(nT)\,\underline{H}\,\underline{x}\,(nT)$$

where u_n is defined by equation 25 as

$$54.\quad \frac{du_n'}{du}\quad \underline{A}_n'\,((n+1)T)\;\underline{K}\left[\underline{y}\,((n+1)T) + \underline{A}_n\,((n+1)T)\;\underline{u}_n\right] = 0$$

For the special case of a linear system R = J = 1 and

$$55.\quad \underline{y}\,((n+1)T) = x_n\,((n+1)T) = \underline{F}\,(nT, (n+1)T)\,\underline{x}(nT),$$

where $\underline{F}$ is the true state transition matrix, while u_n and $\underline{a}$ are defined by equation 35 and 36, so that, with reference to equation 33 and with $\underline{c}\,((n+1)T) = 0$,

$$56.\quad \underline{x}((n+1)T) = \left(\underline{I} + \frac{\underline{a}((n+1)T)\,\underline{a}'\,((n+1)T)\,\underline{K}}{\underline{a}'\,((n+1)T)\,\underline{K}\,\underline{a}\,((n+1)T)}\right)\underline{F}\,(nT, (n+1)T)\underline{x}(nT)$$

Then equation 49 takes the form

$$57.\quad E_{n+1} = -\underline{x}'\,(nT)\;\underline{M}\,\underline{x}\,(nT)$$

where

$$58.\quad \underline{M} = \underline{H} - \underline{F}'\,(nT,\,(n+1)T)\left[\underline{I} + \frac{\underline{K}'\,\underline{a}\,((n-1)T)\,\underline{a}'\,((n-1)T)}{\underline{a}'\,((n-1)T)\underline{K}\,\underline{a}\,((n-1)T)}\right]\underline{H}\left[\underline{I} + \frac{\underline{a}((n-1)T)\underline{a}'((n-1)T)\underline{K}}{\underline{a}'((n-1)T)\underline{K}\underline{a}((n-1)T)}\right]$$
$$\underline{F}(nT,(n+1)T)$$

Then by Liapunov's second method it is sufficient for global asymptotic stability for matrix $\underline{M}$ to be positive definite. Furthermore, for the stationary case where $\underline{F}(nT,(n+1)T) = \underline{F}$ and $\underline{a}(nT) = \underline{a}$, a positive definite $\underline{M}$ will be defined for a positive definite $\underline{H}$ by equation 58 provided the matrix

$$59.\quad \underline{D} = \left[\underline{I} + \frac{\underline{a}\,\underline{a}'\,\underline{K}}{\underline{a}'\,\underline{K}\,\underline{a}}\right]\underline{F}$$

has eigenvalues of absolute value less than one. The experimental results recorded in the sequel indicate that this condition circumscribes well the stable regions of operation.

Since equation 54 is of third order in u_n even for R = 2, no closed form for u_n is practical. Consequently, it is difficult to reach general conclusions for nonlinear plants.

Assuming R = 1, however, limits might be established for the stability of nonlinear plants, with $\underline{c}\,((n+1)T) = 0$ when controlled by the linear, R = 1, control law of equation 35 and 36.

If it is assumed that the system is invariant, then in equation 50 matrix $\underline{A}_n(n+1)T)$ is constant and determined for a specific system by equation 52; so that, with reference to equation 35, the sensitivity $\underline{a}$ becomes

$$60.\quad \underline{a} = \frac{\sum_{j=1}^{J} u_{n-1}^{j}\underline{A}_n j}{u_{n-1}} = \sum_{j=1}^{J} u_{n-1}^{j-1}\,\underline{A}_n j$$

provided $\underline{A}_n j$ are the columns of $\underline{A}_n$.

In other words, the sensitivity $\underline{a}$ that is used is a function of the preceding control force for the R = 1 assumption when the system is actually invariant.

Then the control force is selected by equation 36 as

$$61.\quad u_n = \frac{\sum_{j=1}^{J} (u_{n-1}^{j-1}\, \underline{A}_n j)\, \underline{K}\, \underline{A}}{\sum_{j=1}^{J}\sum_{i=1}^{J} u_{n-1}^{j+i-2}\, \underline{A}_n i\, \underline{K}\, \underline{A}_n j}\, \underline{x}(nT) = \underline{b}'\,(u_{n-1})\underline{x}(nT)$$

Then with equation 50

$$62.\quad \underline{x}\,((n+1)T) = \underline{y}\,((n+1)T) + \sum_{j=1}^{J} \underline{A}_n j \left[\underline{b}'(u_{n-1})\underline{x}(nT)\right]^j$$

and with equation 53

$$63.\quad E = \Delta V(\underline{x}(nT) + \Delta W((\underline{x}(nT), u_{n-1})$$

where

$$64.\quad \Delta W = +2 \sum_{j=1}^{J}\left[\underline{b}'\,(u_{n-1})\underline{x}(nT)\right]^j \left[\underline{y}((n+1)T)\underline{H}\, \underline{A}_n j\right] + \sum_{i=1}^{J}\left[(\underline{b}'(u_{n-1})\underline{x}(nT))^i \underline{A}_n i \underline{H}\, \underline{A}_n j\right]$$

$$65.\quad \Delta V = \underline{y}'((n+1)T)\, \underline{H}\, \underline{y}\,((n+1)T) - \underline{x}'\,(nT)\, \underline{H}\, \underline{x}(nT)$$

Now a sufficient condition for stability is, by Liapunov's second method, that

$$66.\quad \Delta W + \Delta V < 0 \quad \text{for } \underline{H} = \underline{I} \text{ for all } \underline{x}(nT) \text{ and all } u_{n-1}$$

Considering that $\underline{y}\,((n+1)T)$, the free response starting at $t = nT$ from state $\underline{x}(nT)$, depends only on $\underline{x}(nT)$ if the system is stable without the control, it may have a Liapunov function of the form $V(x) = \|x\|^2$; then ΔV, as given in equation 65, is negative definite for $\underline{H} = \underline{I}$. So it is enough for ΔW to be nonpositive to show that the controlled system is stable. This may be shown for specific systems but the fact that u_{n-1} is present in equation 64 causes additional difficulties. u_{n-1} itself is varying during the trajectory and is defined by equations in the nature of equation 61 which are too complex to permit exact consideration. Since u is constrained to $|u| \leq U$, it may be possible to show ΔW to be nonpositive for all u values within this constraint. The system then would be proved stable but the condition is rather restrictively sufficient.

III. HYBRID SIMULATION (FIFTH ORDER)

Introduction

The following system was selected for the hybrid simulation study:

$$G(s) = \frac{1}{s\left[(s+0.5)^2+1\right]\left[(s+6)^2+5^2\right]}$$

This system was taken from the fifth order systems previously investigated in the all-digital simulation study. It was simulated on the analog computer with a desired set of initial conditions for C (position), $\dot{C}$ (velocity), $\ddot{C}$ (acceleration), $\dddot{C}$, and $\ddddot{C}$. A hybrid run consisted of driving the system from the position representing the inital conditions to zero with the signals from the digital computer.

The hybrid simulation runs included in this section are presented to illustrate the effect of using a finite Taylor Polynomial to approximate the state transition matrix, and to illustrate the effect of filtering and prediction to estimate the state variables from sampled data. The runs were made for different decision interval times (T), and for different weighting factors (h). In all cases the values of T and h were selected from the performance and stability boundaries for this system. These boundaries were included in the last quarterly progress report. The estimated unit step response, $a_i(0)] = T^i/i!]$ (fifth order system; i = 5, 4, 3, 2, 1) was held constant throughout each run. Also, all computations were made in floating point arithmetic.

The runs, made with sampled (exact) state variables, obtained these variables directly from the analog computer. The runs, with estimated state variables, obtained them by filtering and prediction from sampled data. In the hybrid simulations with filtering, the fitting matrix that was used was derived from early prediction studies. All hybrid runs used least square polynomial fitting, fourth degree polynomial fit, and a sample interval of 0.15 seconds. Also, eight samples of data were used in all cases.

A. Sampled State Variables - Exact Runs

Hybrid runs were made for the system using sampled (exact) state variables, and the exact state transition matrix from the all digital study. These runs were then compared with the all digital runs to confirm the accuracy of the simulation and results. These runs are equivalent except for such effects as A-D and D-A conversion, computational round off, and a slightly different time origin, which are taken into account with the hybrid simulation. The maximum available force in the hybrid simulations was $\pm 10^2$ units, whereas in the all digital simulations the force was $\pm 10^4$ units. The results were satisfactory, and closely approached the all digital results in smoothness and settling time. Typical results are plotted in Figures 1, 2, 3, and 4.

B. Sampled State Variables - Taylor Runs

Figures 5, 6, and 7, are comparisons of hybrid and all digital runs with known state variables and Taylor approximate state transition matrix. These figures demonstrate the same conclusions as the previous paragraph.

The hybrid runs with sampled state variables and the Taylor approximate state transition matrix were run to study truncation error. The truncation error is the error caused by using a finite Taylor Polynomial to approximate the state transition matrix. Typical results of this study are presented in Figures 9, 10, and 11. In all cases, the truncation error appears negligible, since settling time and smoothness of response are still adequate for good control.

A series of runs were performed with this system to see the effect of using different initial conditions. The resulting responses are presented in Figures 13 and 14. The settling time and amount of overshoot differed for each set of initial conditions, but the system exhibited satisfactory control in each case.

Certain common nonlinearities were studied by another series of hybrid runs. Figure 15 demonstrates the effect of a deadzone in the applied force, and Figures 16 and 17 show the effect of position, velocity, and acceleration saturation. The results indicate that adequate control was maintained, and so the results can be judged satisfactory.

One last area was investigated with hybrid runs with sampled state variables. If the actual order of a system is very large, it is likely to be controlled for a lower than actual order. Figure 8 shows the effect of controlling a fifth order system as if it were a fourth or third order system. Successful control was possible in both cases, and the control did not deteriorate in any objectionable degree.

C. Estimated State Variables - Taylor Runs

Hybrid runs were made for the system using estimated state variables, and Taylor approximate state transition matrix. These runs were made to study the effect of filtering and prediction to estimate the state variables from sampled data. Typical results are illustrated by Figures 18, 19, and 20. These graphs compare the hybrid runs with sampled state variables and the hybrid runs with estimated state variables. Exact comparison was not possible because the sampled state variable runs outputed a programmed initial force for three decision intervals, whereas the estimated state variable runs outputed a programmed initial force for a different number of intervals. The results were satisfactory, but usually not as good as those obtained with the sampled state variables.

A series of runs was made using different initial conditions. Figure 21 shows some of the results of these runs. Also, Figure 12 compares the hybrid sampled and estimated state variable runs for a dead zone in the applied force. The control illustrated in both figures is satisfactory, but not quite as good as that obtained when the state variables were directly sampled from the analog computer.

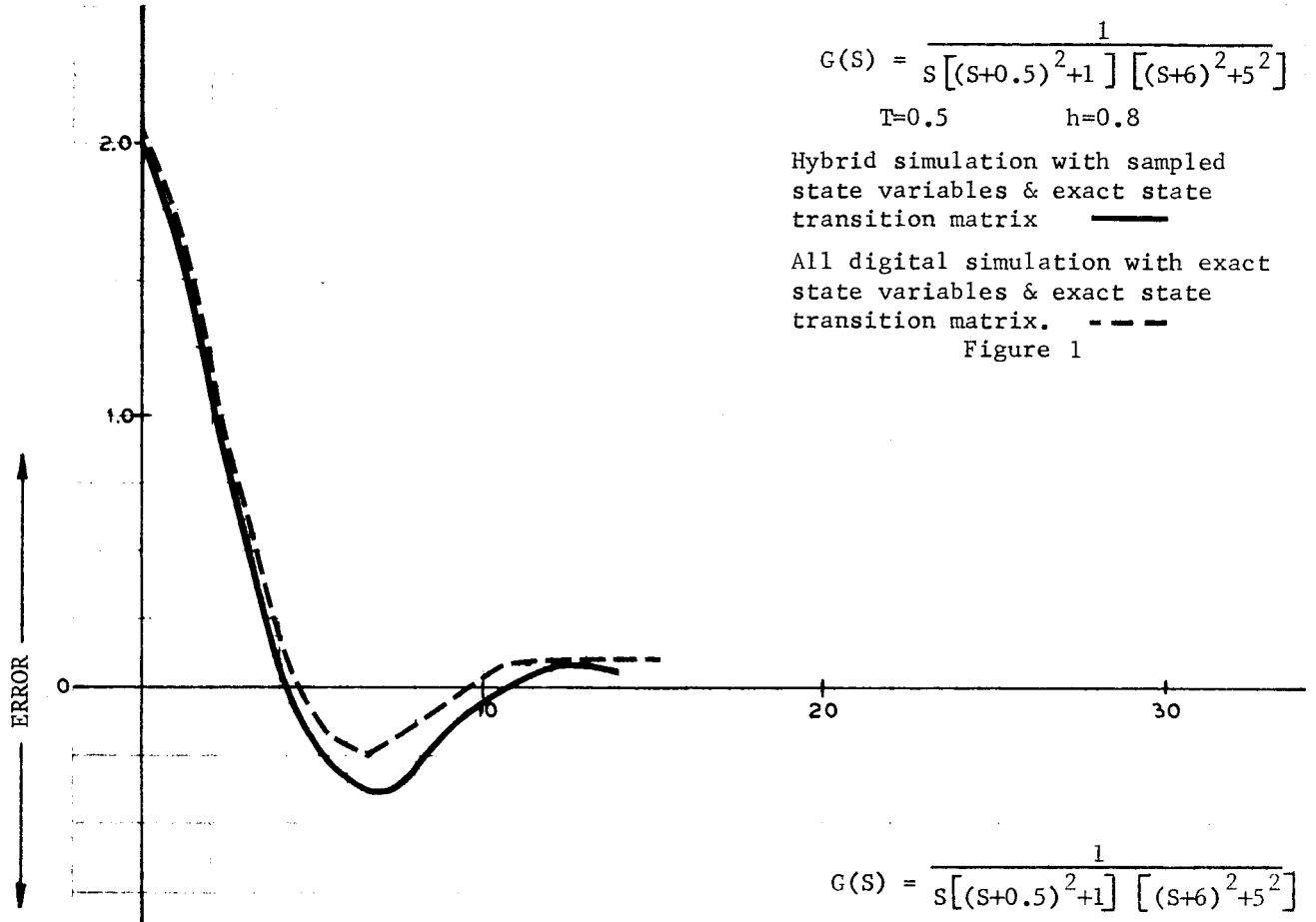
$$G(S) = \frac{1}{S\left[(S+0.5)^2+1\right]\left[(S+6)^2+5^2\right]}$$

T=0.5 h=0.8

Hybrid simulation with sampled state variables & exact state transition matrix ——

All digital simulation with exact state variables & exact state transition matrix. - - -

Figure 1



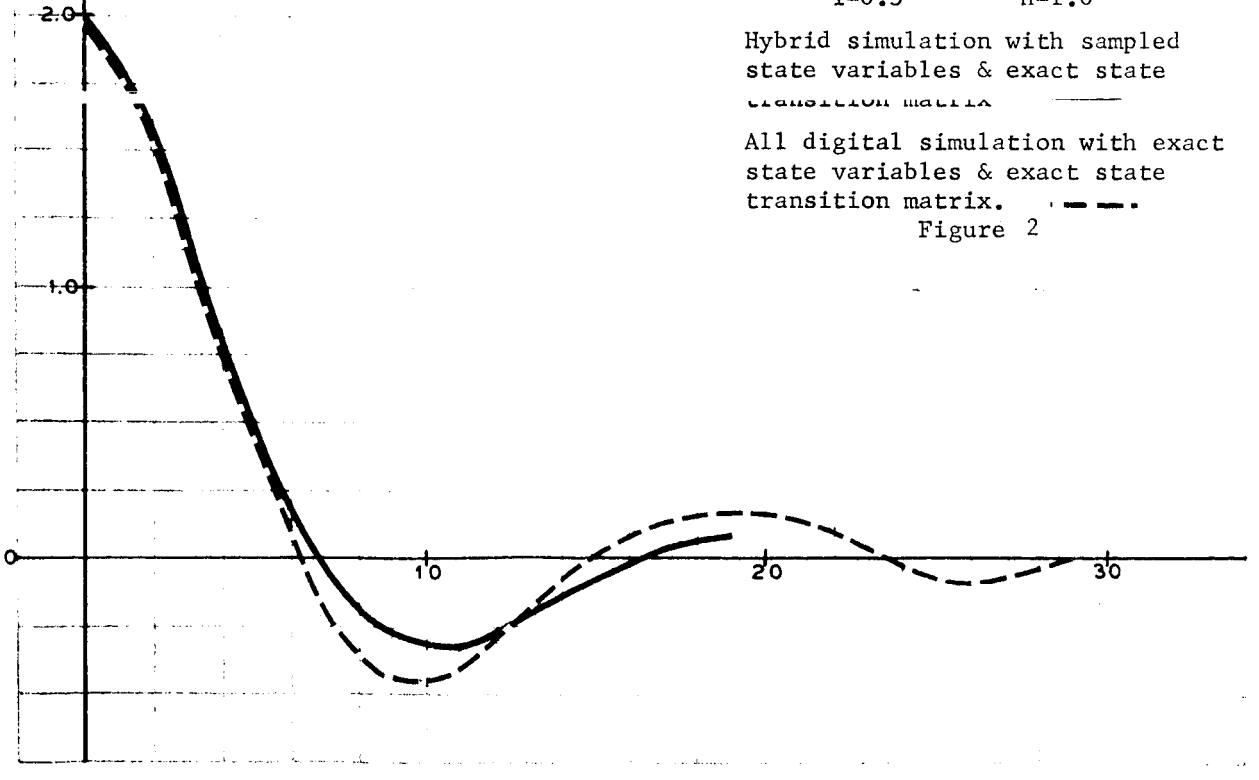
$$G(S) = \frac{1}{S\left[(S+0.5)^2+1\right]\left[(S+6)^2+5^2\right]}$$

T=0.3 h=1.0

Hybrid simulation with sampled state variables & exact state transition matrix ——

All digital simulation with exact state variables & exact state transition matrix. - · - · -

Figure 2



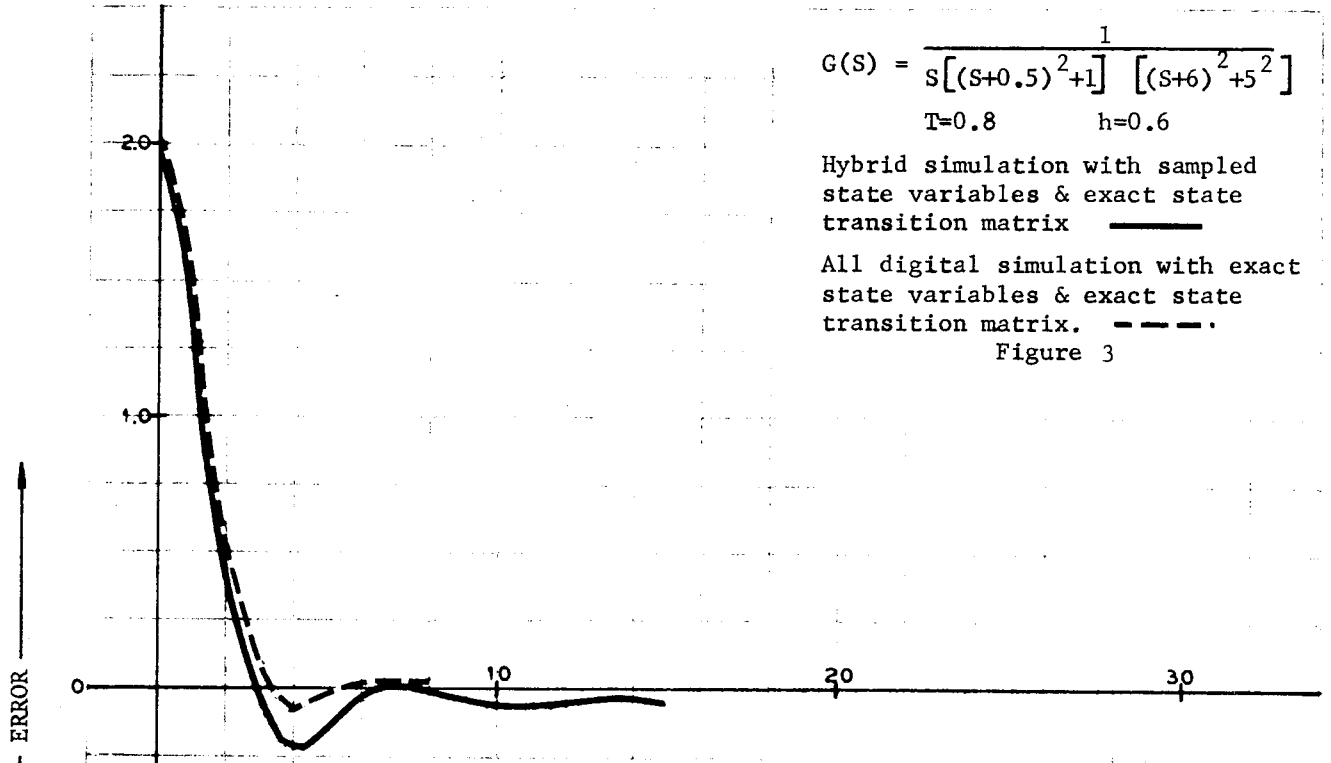
$$G(S) = \frac{1}{S\left[(S+0.5)^2+1\right]\left[(S+6)^2+5^2\right]}$$

T=0.8 h=0.6

Hybrid simulation with sampled state variables & exact state transition matrix ——

All digital simulation with exact state variables & exact state transition matrix. — — —

Figure 3



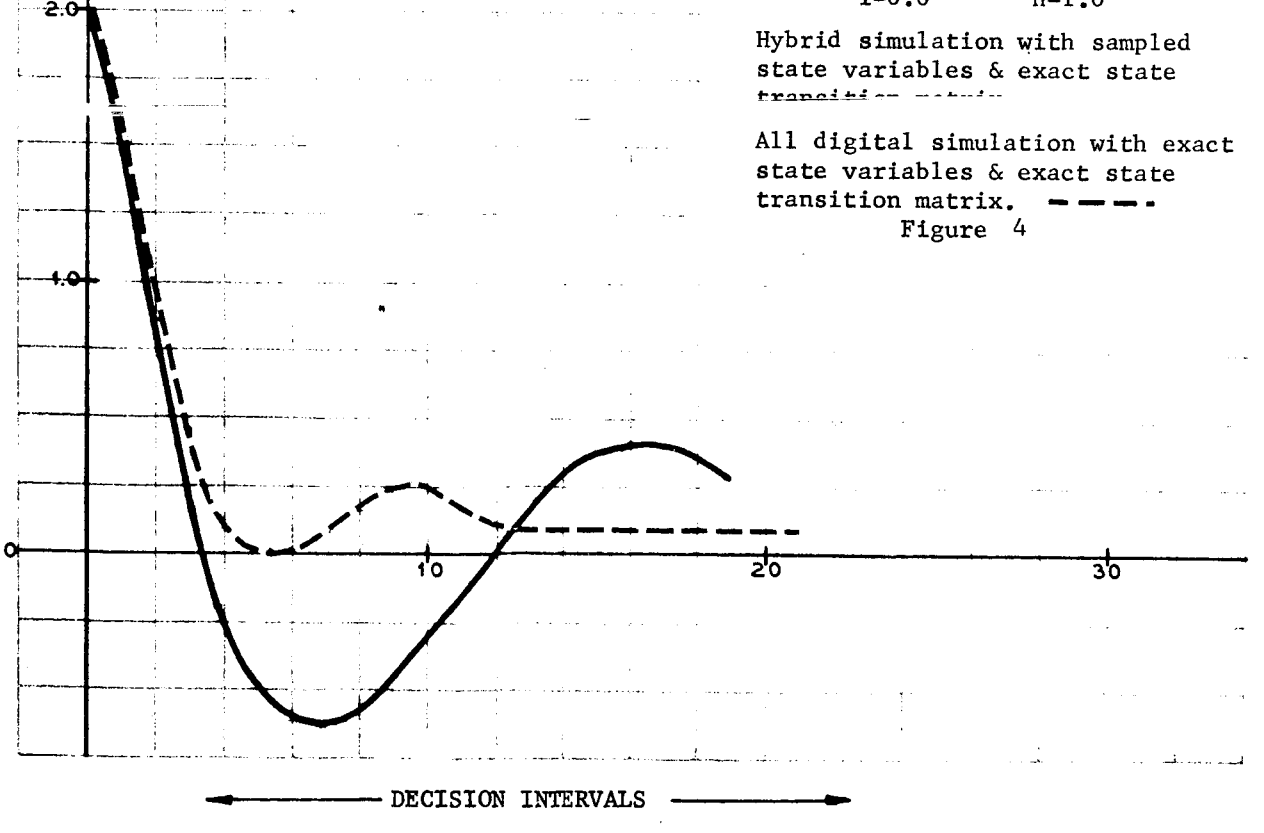
$$G(S) = \frac{1}{S\left[(S+0.5)^2+1\right]\left[(S+6)^2+5^2\right]}$$

T=0.6 h=1.0

Hybrid simulation with sampled state variables & exact state transition matrix

All digital simulation with exact state variables & exact state transition matrix. — — —

Figure 4



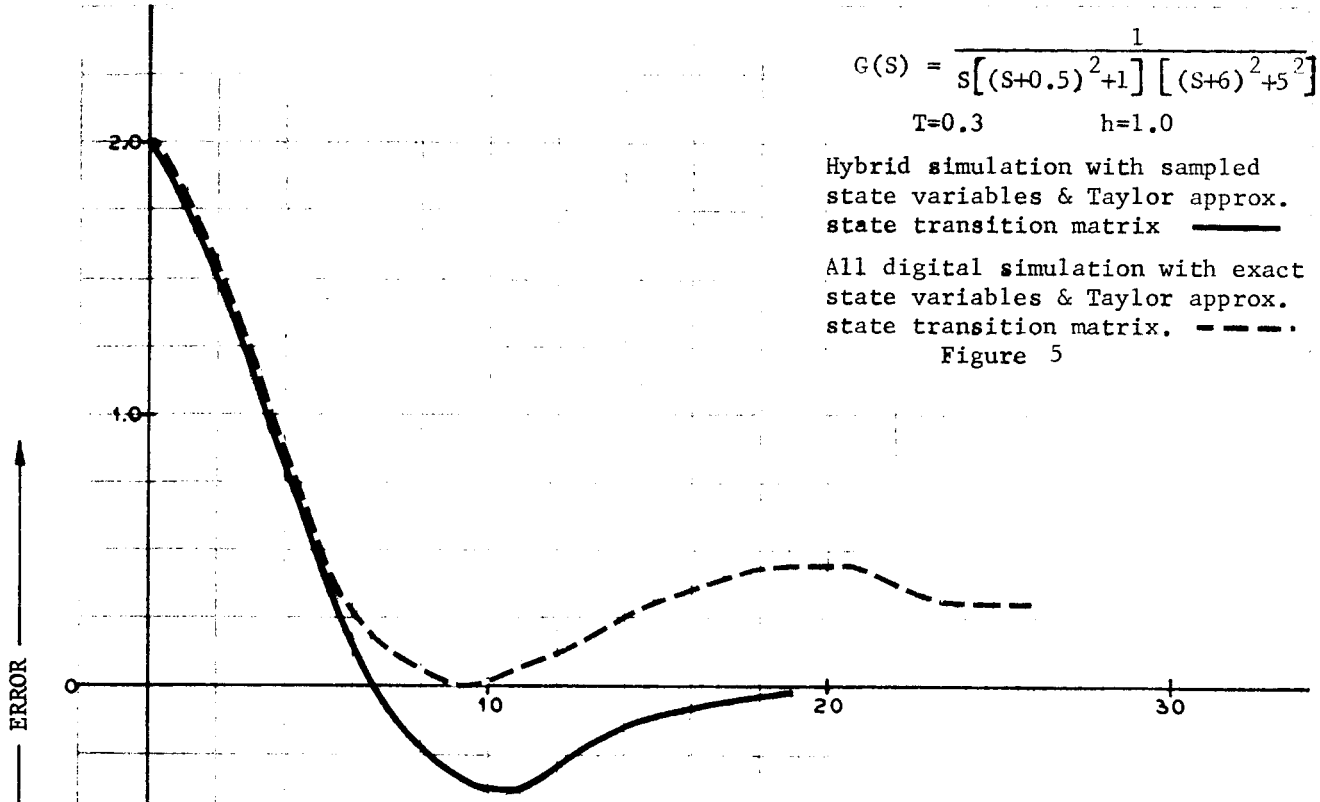
$$G(S) = \frac{1}{S\left[(S+0.5)^2+1\right]\left[(S+6)^2+5^2\right]}$$

T=0.3 h=1.0

Hybrid simulation with sampled state variables & Taylor approx. state transition matrix ——

All digital simulation with exact state variables & Taylor approx. state transition matrix. - - - -

Figure 5



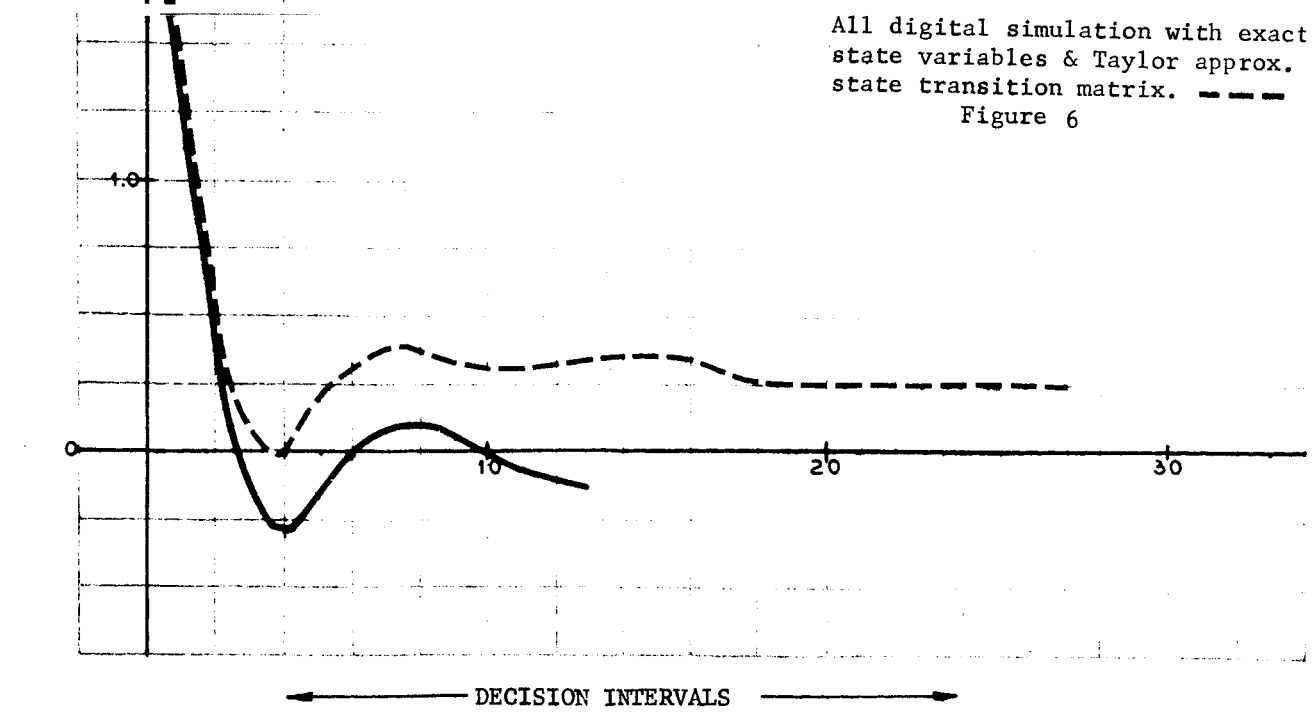
$$G(S) = \frac{1}{S\left[(S+0.5)^2+1\right]\left[(S+6)^2+5^2\right]}$$

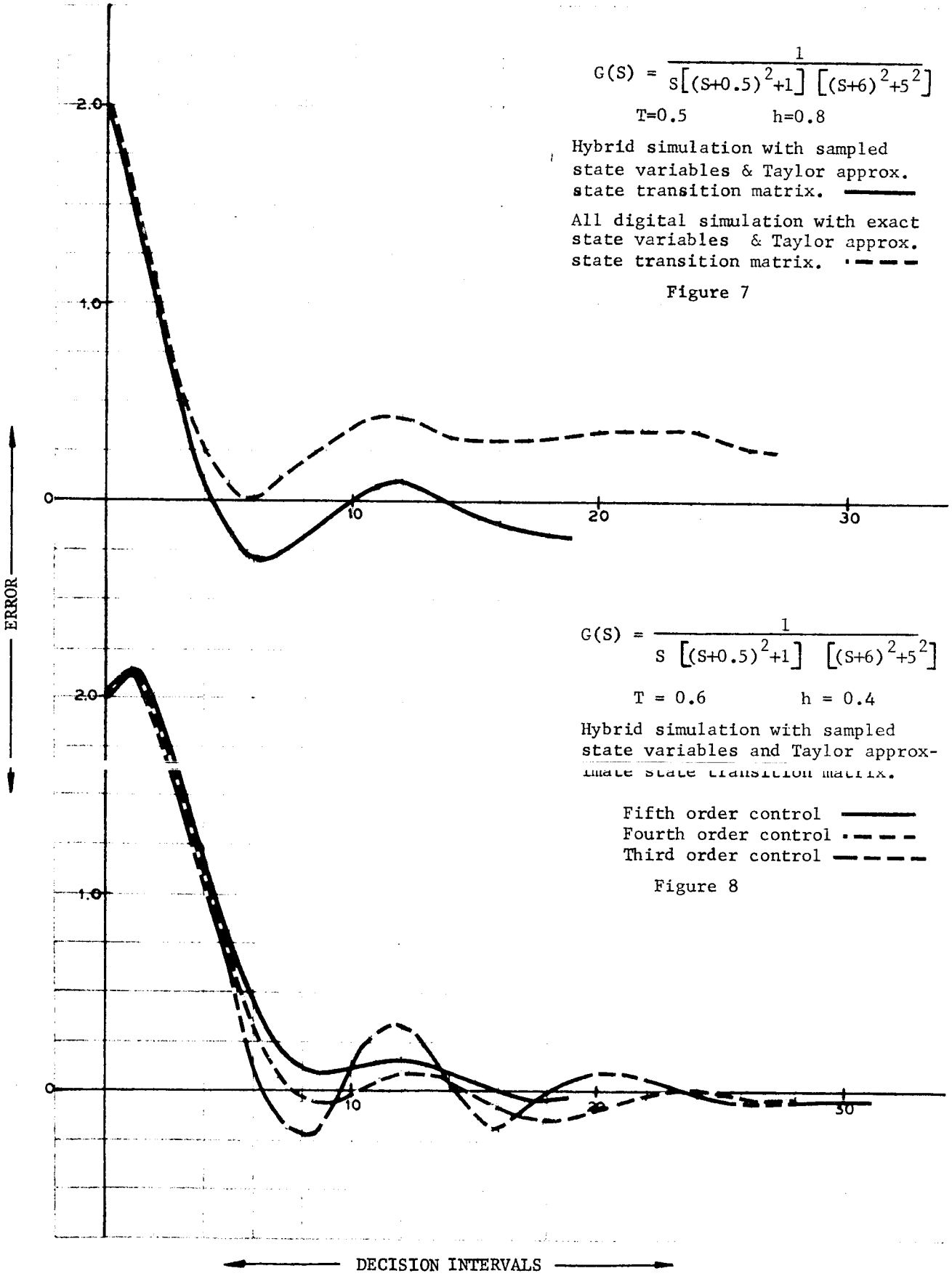
T=0.8 h=0.6

Hybrid simulation with sampled state variables & Taylor approx. state transition matrix ——

All digital simulation with exact state variables & Taylor approx. state transition matrix. - - -

Figure 6





$$G(S) = \frac{1}{S\left[(S+0.5)^2+1\right]\left[(S+6)^2+5^2\right]}$$

T=0.5 h=0.8

Hybrid simulation with sampled state variables & Taylor approx. state transition matrix. ——

All digital simulation with exact state variables & Taylor approx. state transition matrix. ·— — —

Figure 7

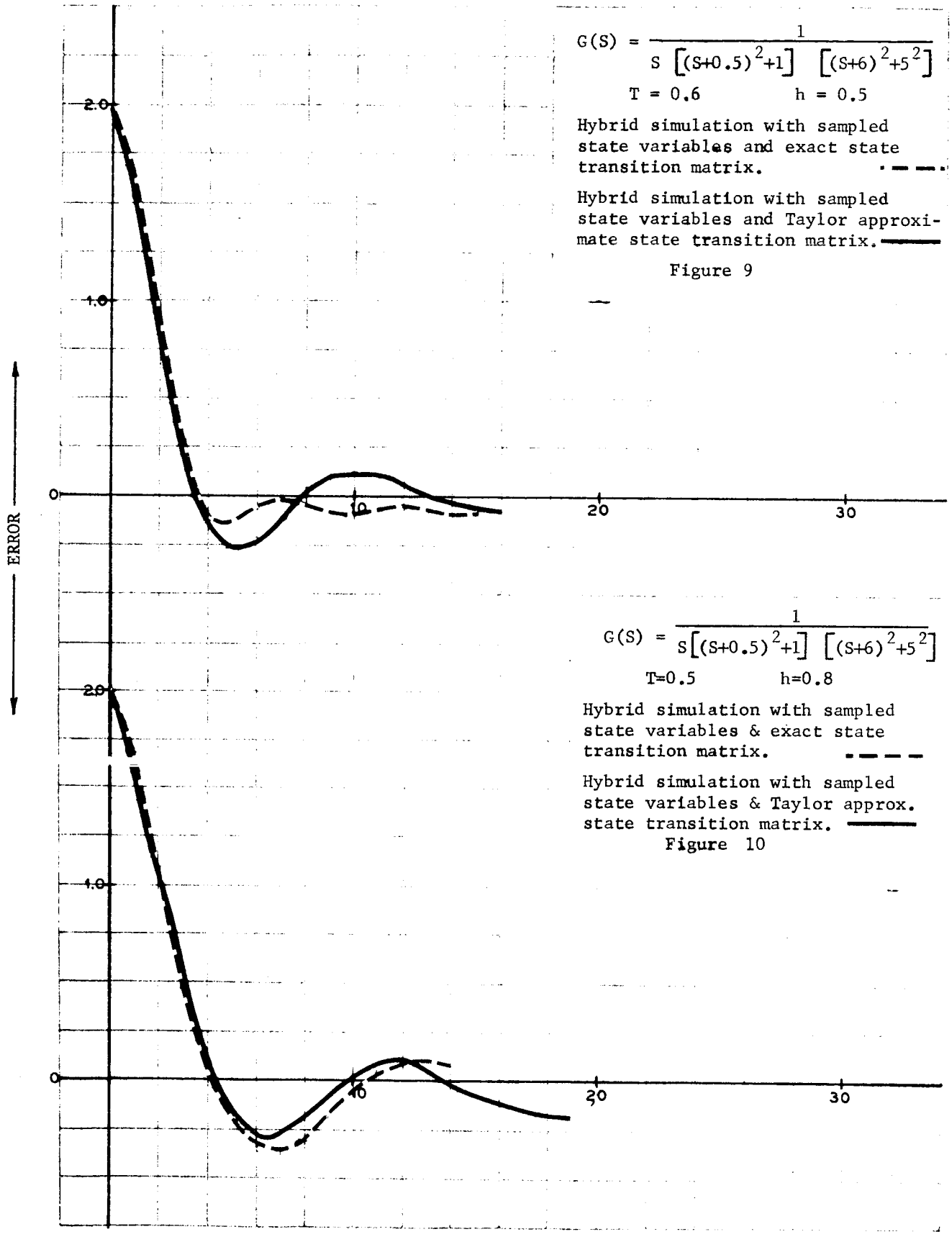
$$G(S) = \frac{1}{S\left[(S+0.5)^2+1\right]\left[(S+6)^2+5^2\right]}$$

T = 0.6 h = 0.4

Hybrid simulation with sampled state variables and Taylor approximate state transition matrix.

Fifth order control ——
Fourth order control ·— — —
Third order control — — — —

Figure 8



$$G(S) = \frac{1}{S\left[(S+0.5)^2+1\right]\left[(S+6)^2+5^2\right]}$$

T = 0.6 h = 0.5

Hybrid simulation with sampled state variables and exact state transition matrix. - - -

Hybrid simulation with sampled state variables and Taylor approximate state transition matrix. ———

Figure 9

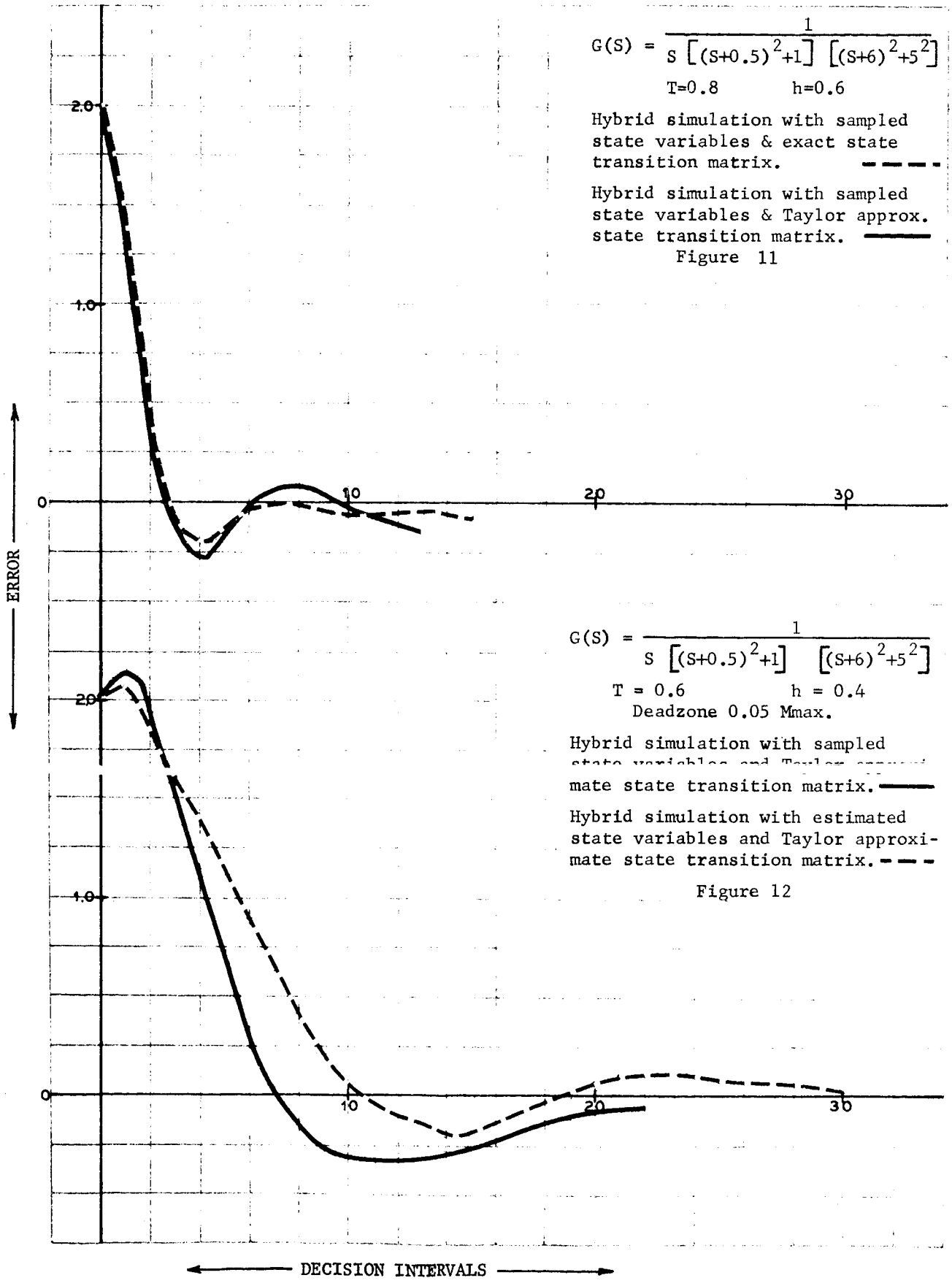
$$G(S) = \frac{1}{S\left[(S+0.5)^2+1\right]\left[(S+6)^2+5^2\right]}$$

T=0.5 h=0.8

Hybrid simulation with sampled state variables & exact state transition matrix. - - - -

Hybrid simulation with sampled state variables & Taylor approx. state transition matrix. ———

Figure 10



$$G(S) = \frac{1}{S\left[(S+0.5)^2+1\right]\left[(S+6)^2+5^2\right]}$$

T=0.8 h=0.6

Hybrid simulation with sampled state variables & exact state transition matrix. - - - -

Hybrid simulation with sampled state variables & Taylor approx. state transition matrix. ——

Figure 11

$$G(S) = \frac{1}{S\left[(S+0.5)^2+1\right]\left[(S+6)^2+5^2\right]}$$

T = 0.6 h = 0.4

Deadzone 0.05 Mmax.

Hybrid simulation with sampled state variables and Taylor approximate state transition matrix. ——

Hybrid simulation with estimated state variables and Taylor approximate state transition matrix. - - -

Figure 12

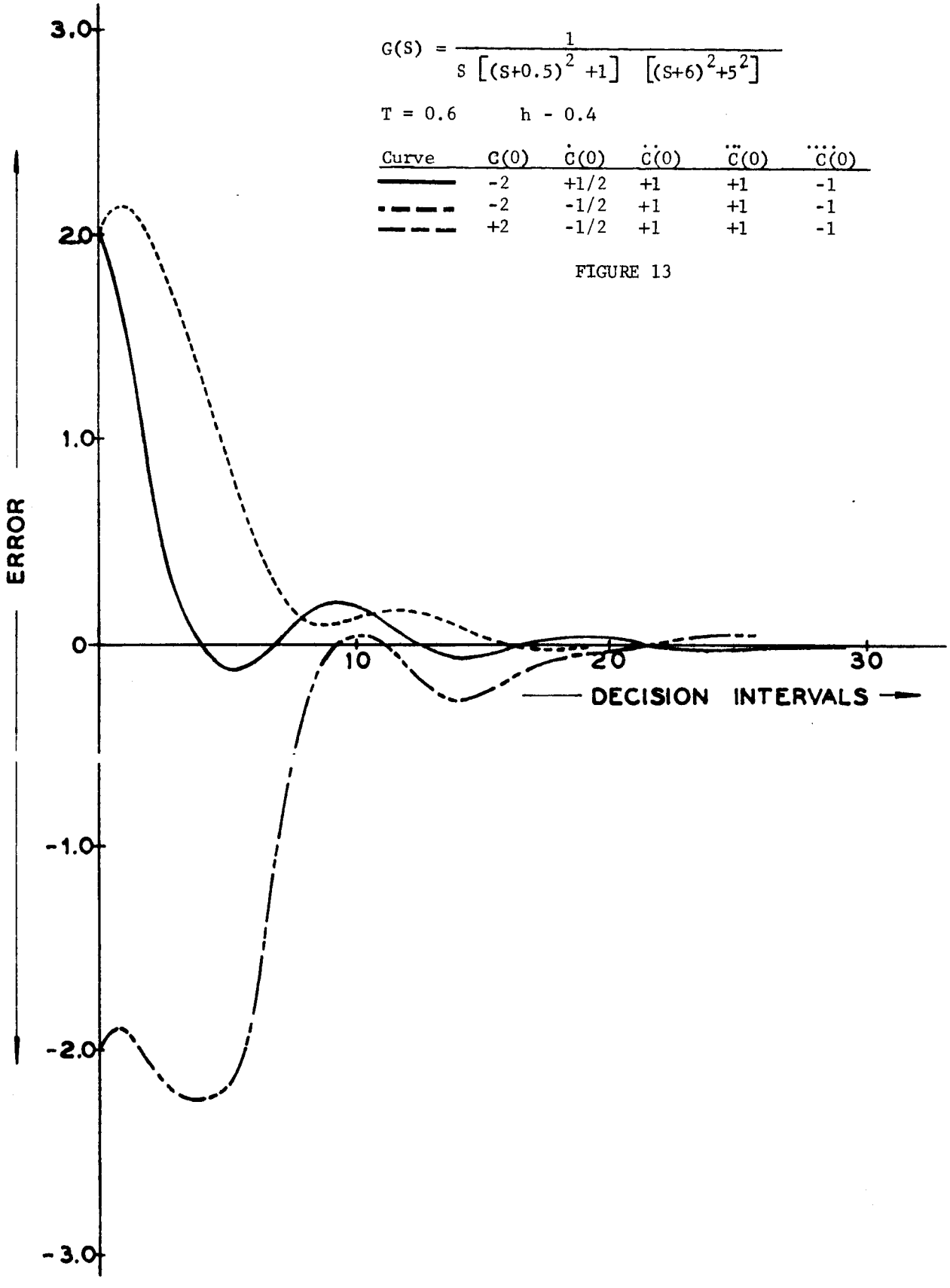
Hybrid simulation with sampled state variables and Taylor approximate state transition matrix

$$G(S) = \frac{1}{S\left[(S+0.5)^2+1\right]\left[(S+6)^2+5^2\right]}$$

T = 0.6 h - 0.4

Curve	$C(0)$	$\dot{C}(0)$	$\ddot{C}(0)$	$\dddot{C}(0)$	$\ddddot{C}(0)$
———	-2	+1/2	+1	+1	-1
- - - -	-2	-1/2	+1	+1	-1
— — —	+2	-1/2	+1	+1	-1

FIGURE 13



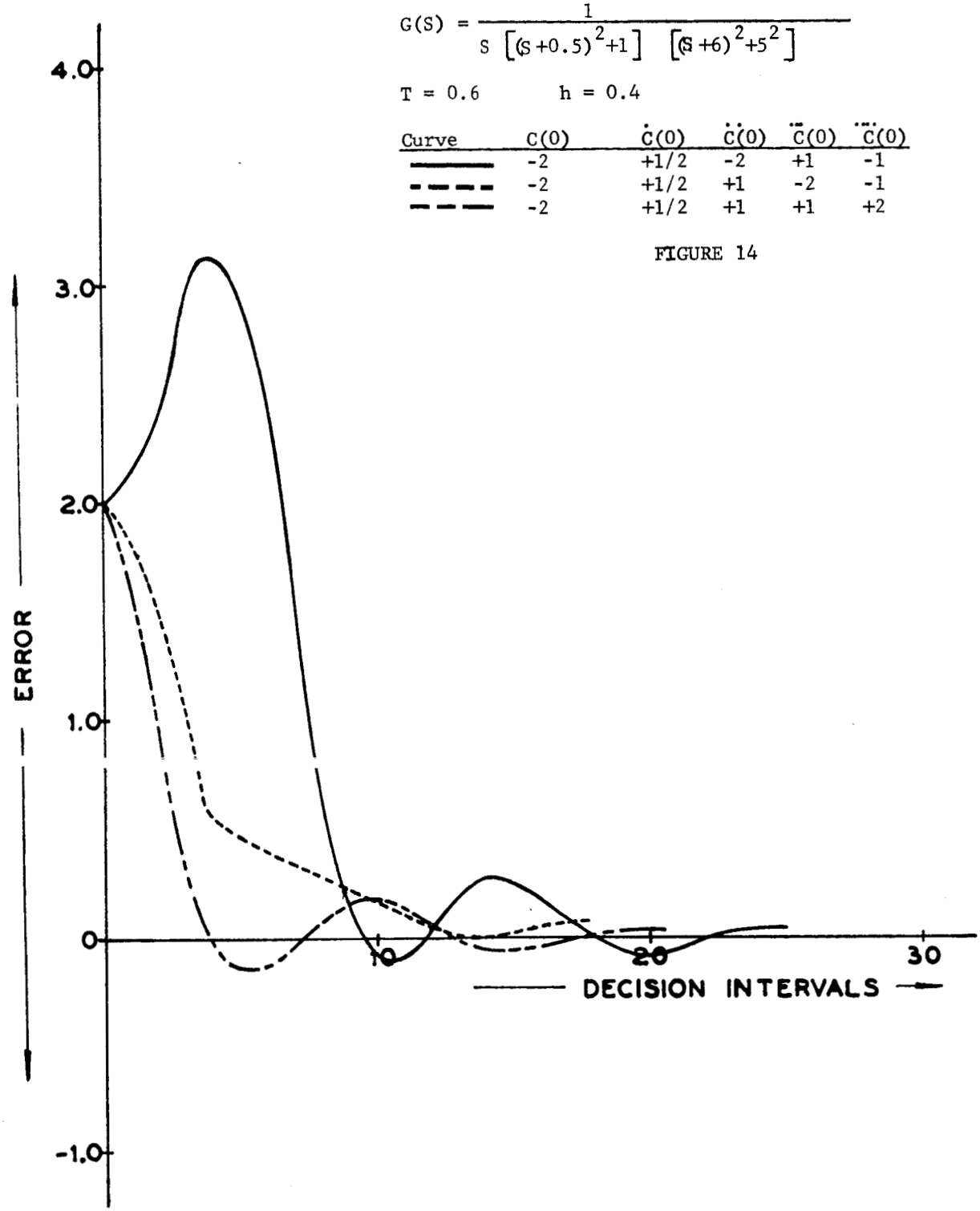
Hybrid simulation with sampled state variables and Taylor approximate state transition matrix

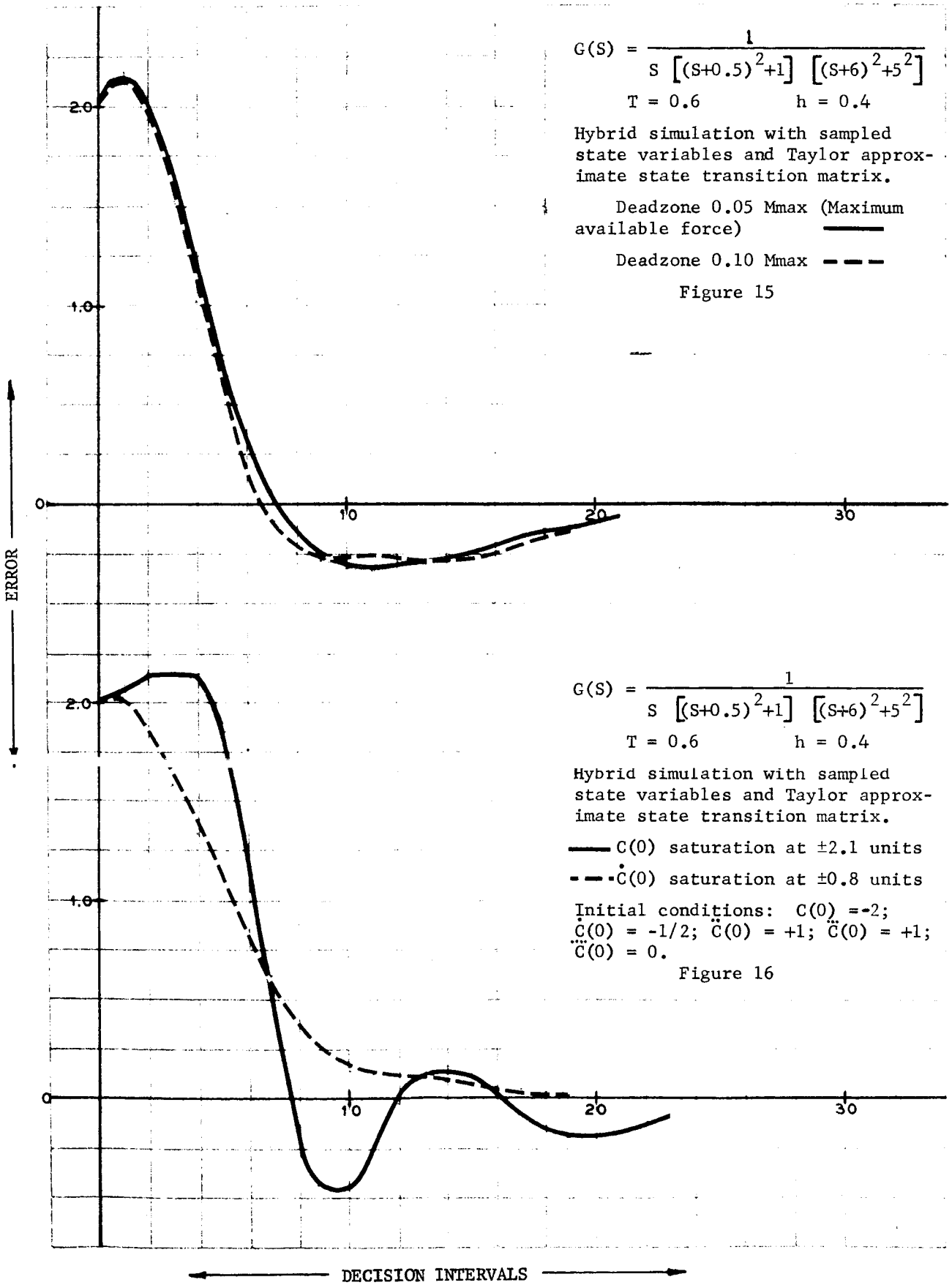
$$G(S) = \frac{1}{S\left[(S+0.5)^2+1\right]\left[(S+6)^2+5^2\right]}$$

T = 0.6 h = 0.4

Curve	C(0)	$\dot{C}(0)$	$\ddot{C}(0)$	$\dddot{C}(0)$	$\ddddot{C}(0)$
———	-2	+1/2	-2	+1	-1
- - - -	-2	+1/2	+1	-2	-1
— — —	-2	+1/2	+1	+1	+2

FIGURE 14





$$G(S) = \frac{1}{S\left[(S+0.5)^2+1\right]\left[(S+6)^2+5^2\right]}$$

T = 0.6 h = 0.4

Hybrid simulation with sampled state variables and Taylor approximate state transition matrix.

Deadzone 0.05 Mmax (Maximum available force) ——

Deadzone 0.10 Mmax - - -

Figure 15

$$G(S) = \frac{1}{S\left[(S+0.5)^2+1\right]\left[(S+6)^2+5^2\right]}$$

T = 0.6 h = 0.4

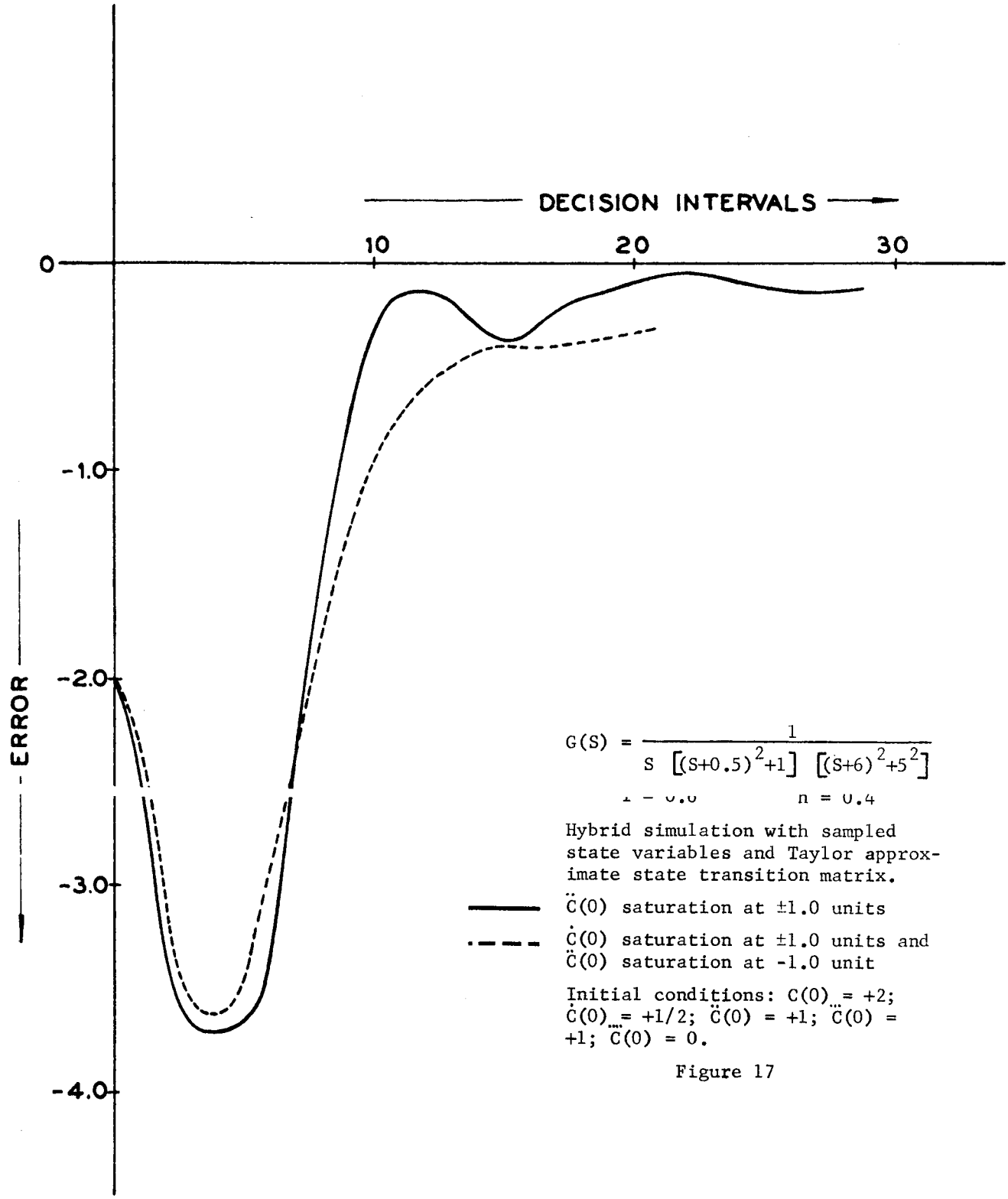
Hybrid simulation with sampled state variables and Taylor approximate state transition matrix.

—— C(0) saturation at ±2.1 units

- - - $\dot{C}(0)$ saturation at ±0.8 units

Initial conditions: $C(0) = -2$; $\dot{C}(0) = -1/2$; $\ddot{C}(0) = +1$; $\dddot{C}(0) = +1$; $\ddddot{C}(0) = 0$.

Figure 16



$$G(S) = \frac{1}{S\left[(S+0.5)^2+1\right]\left[(S+6)^2+5^2\right]}$$

$\zeta = 0.6$ $\quad$ $n = 0.4$

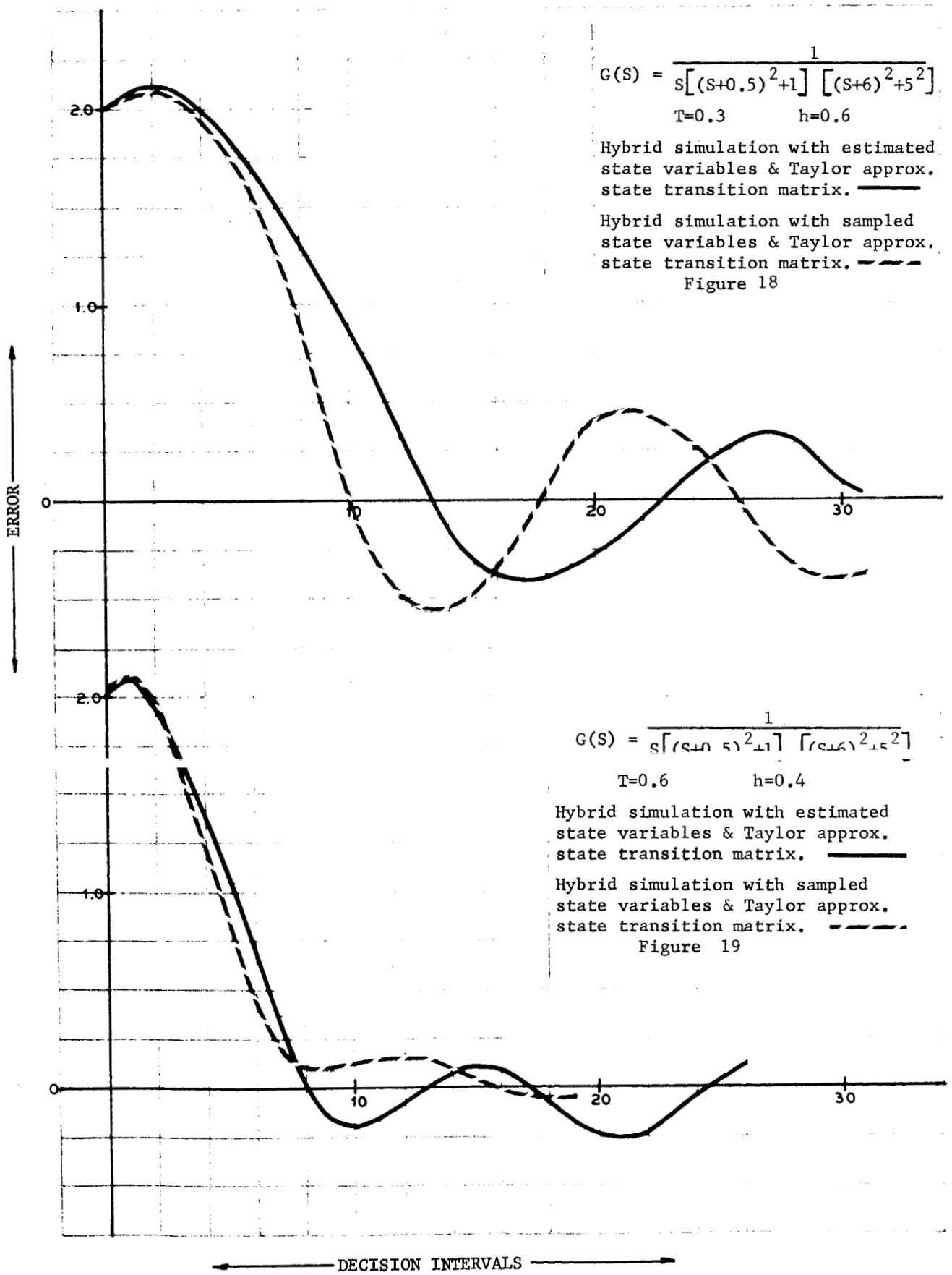
Hybrid simulation with sampled state variables and Taylor approximate state transition matrix.

$\ddot{C}(0)$ saturation at ±1.0 units

$\dot{C}(0)$ saturation at ±1.0 units and $\ddot{C}(0)$ saturation at -1.0 unit

Initial conditions: $C(0) = +2$; $\dot{C}(0) = +1/2$; $\ddot{C}(0) = +1$; $\dddot{C}(0) = +1$; $\ddddot{C}(0) = 0$.

Figure 17



$$G(S) = \frac{1}{S\left[(S+0.5)^2+1\right]\left[(S+6)^2+5^2\right]}$$

T=0.3 h=0.6

Hybrid simulation with estimated state variables & Taylor approx. state transition matrix. ——

Hybrid simulation with sampled state variables & Taylor approx. state transition matrix. — — —

Figure 18

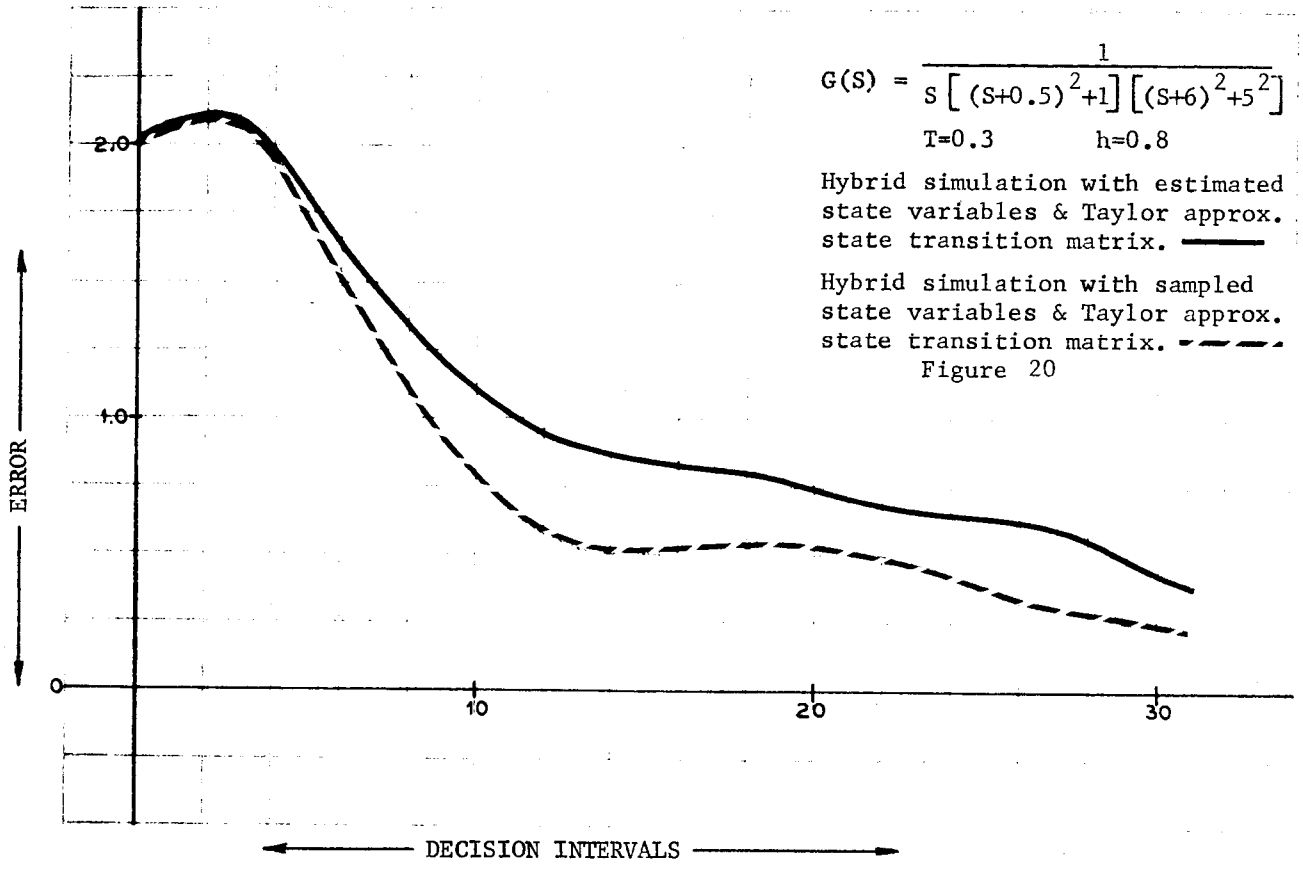
$$G(S) = \frac{1}{S\left[(S+0.5)^2+1\right]\left[(S+6)^2+5^2\right]}$$

T=0.6 h=0.4

Hybrid simulation with estimated state variables & Taylor approx. state transition matrix. ——

Hybrid simulation with sampled state variables & Taylor approx. state transition matrix. — — —

Figure 19



$$G(S) = \frac{1}{S\left[(S+0.5)^2+1\right]\left[(S+6)^2+5^2\right]}$$

T=0.3 h=0.8

Hybrid simulation with estimated state variables & Taylor approx. state transition matrix. ——

Hybrid simulation with sampled state variables & Taylor approx. state transition matrix. - — — -

Figure 20

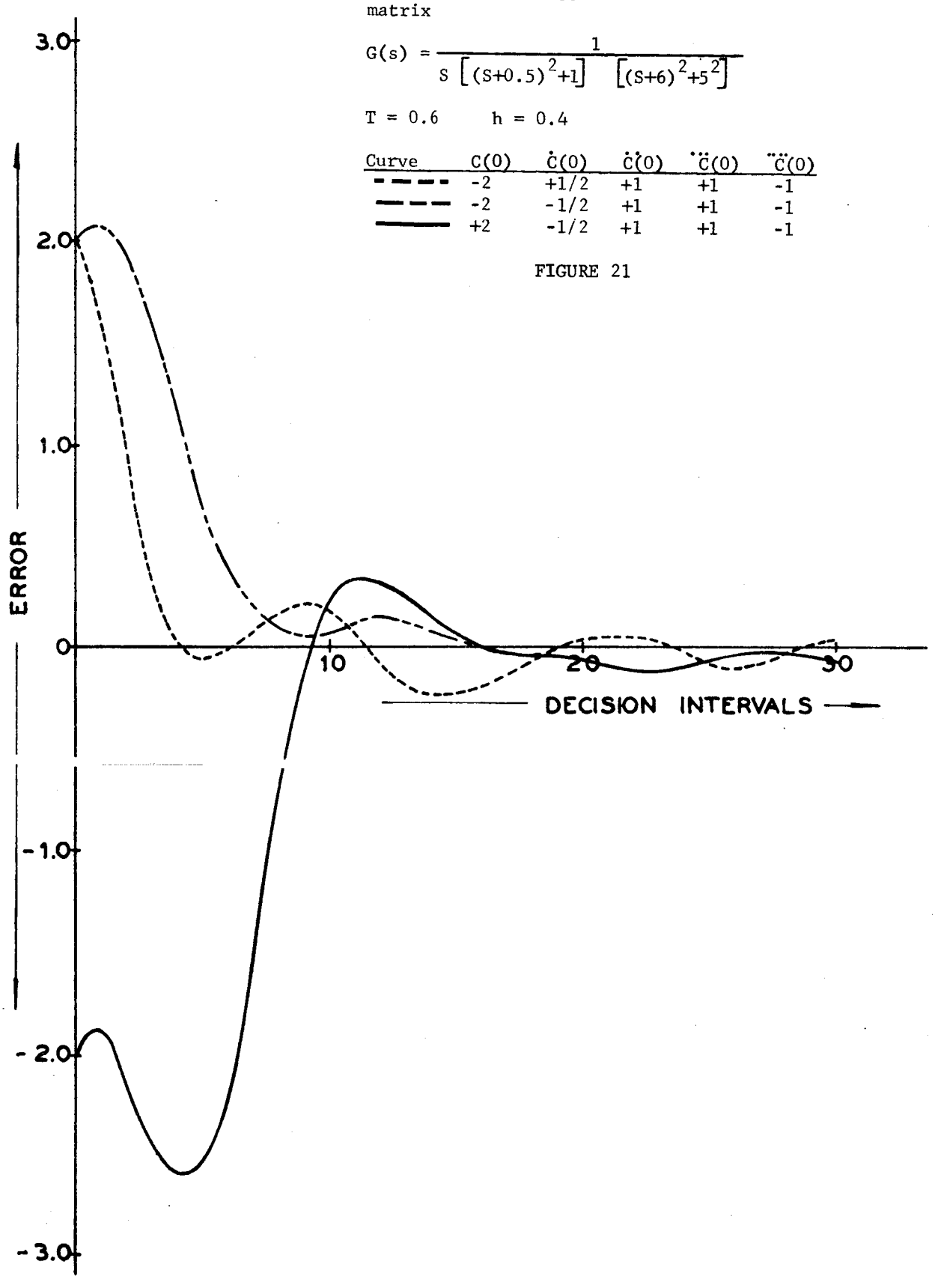
Hybrid simulation with estimated state variables and Taylor approximate state transition matrix

$$G(s) = \frac{1}{S\left[(S+0.5)^2+1\right]\left[(S+6)^2+5^2\right]}$$

T = 0.6 h = 0.4

Curve	C(0)	$\dot{C}(0)$	$\ddot{C}(0)$	$\dddot{C}(0)$	$\ddddot{C}(0)$
- - - -	-2	+1/2	+1	+1	-1
— — —	-2	-1/2	+1	+1	-1
———	+2	-1/2	+1	+1	-1

FIGURE 21



IV. COMPUTER REQUIREMENTS

Introduction

Four major programs for hybrid simulation on the Emerson EM-5000 Digital Computer have been written and have been included in past quarterly progress reports. The last of these programs was chosen for this section, since it was the only program which included the estimation of the state variables.

Hybrid Simulation Program Description

This hybrid simulation program is written in floating point arithmetic, and is good for plants of fifth or lower order. The estimated state variables are obtained by filtering and prediction from eight past values of the response, c(t). These eight past values of c(t) may be obtained in one, two, or four decision intervals; i.e. eight, four or two values of c(t) may be sampled per interval. Also, this program has the estimate of the unit step response, $a_i(0)$, as an input, and so it is fixed for all computations throughout the entire run.

The block diagram for this program is presented in Figure 22. This diagram is identical to the one included in a previous report, except for the addition of the estimated unit step response block. This block was added to the diagram since it most certainly would be incorporated in future programs.

This particular program uses 800 storage locations, which does not include constant locations and subrontine storage. Of the 800 storage locations 227 are the floating point arithmetic instruction locations. The addition of the unit step response block would add about 20 floating point instructions. The following is a list of floating point arithmetic instructions which are used by this program:

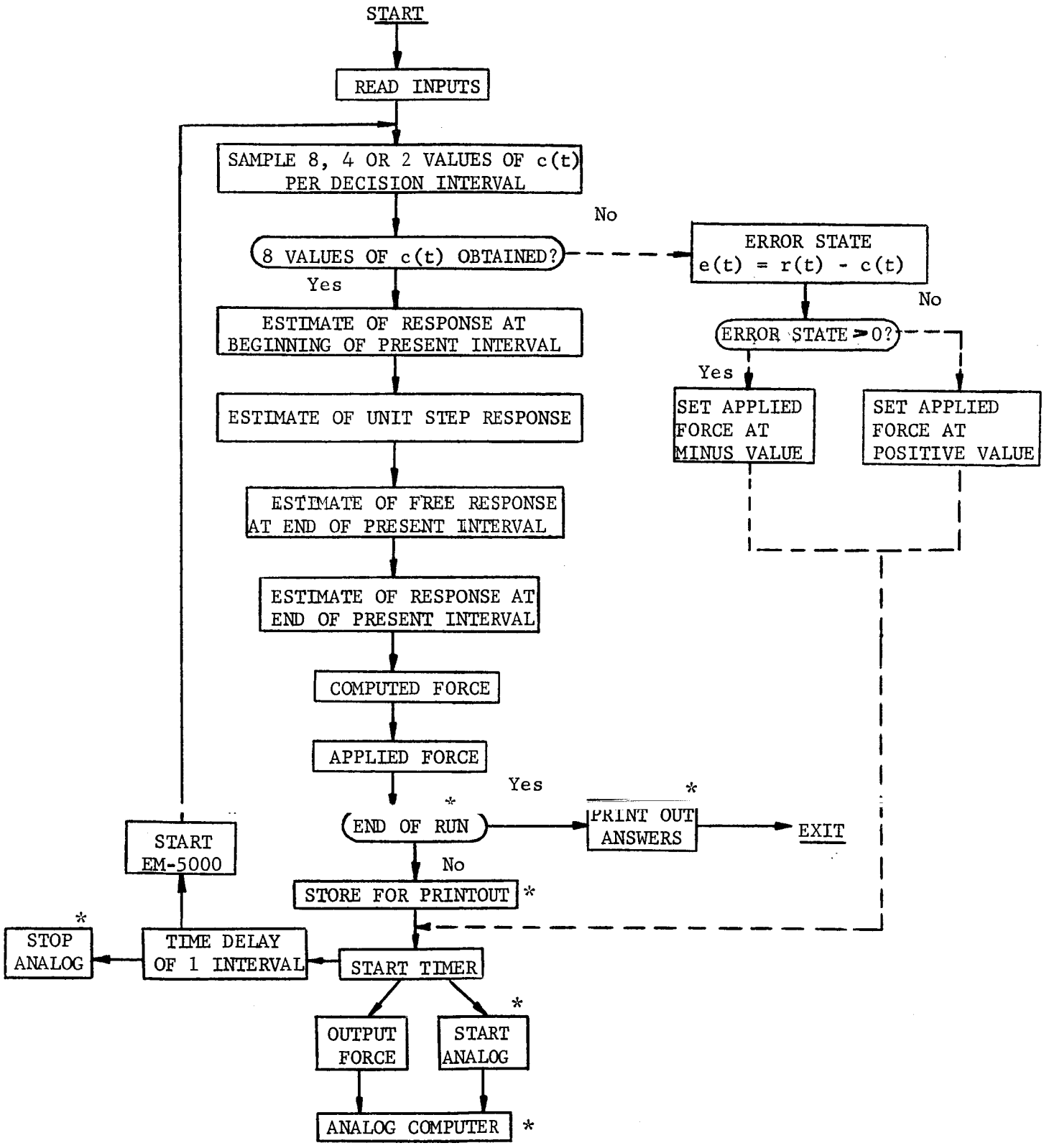
Number of Add Instructions	92
Number of Subtract Instructions	2
Number of Multiply Instructions	127
Number of Divide Instructions	6

Specifications for Applicable Computer

The only points considered in selecting an applicable digital computer were operating speed and programming ease. One computer which could be used is the DDP-24. This is a general purpose digital computer manufactured by the Computer Control Company, Inc. Partial DDP-24 Spacifications follow:

TYPE:

Binary, core memory, parallel, single address with indexing, and indirect addressing.



Note: Functions of starred blocks are peculiar to the hybrid simulator, and need not be realized in an actual control system.

FIGURE 22 HYBRID SIMULATION DIAGRAM

WORD LENGTH:
24 bit; sign/magnitude code.

SPEED: (including instruction and operand access)

Add	10 usec
Multiply	31 usec
Divide	33 usec
Add floating point	
24 bit mantissa, 9 bit characteristic	116 usec
39 bit mantissa, 9 bit characteristic	323 usec
Multiple floating point	
24 bit mantissa, 9 bit characteristic	97 usec
39 bit mantissa, 9 bit characteristic	376 usec
Add double precision fixed point	70 usec
Multiply double precision fixed point	204 usec
1/0 word transfer	5 usec
1/0 block transfer	166,000 words/sec.

MEMORY:
4,096 words, expandable to 16,384; all words addressable; coincident current ferrite core; non-volatile storage. 5 usec cycle time, 3 usec access time. As a special option directly addressable memory expansion to 32,768 words is possible.

Estimation of Program Execution Time Using DDP-24 Computer

The floating point operations required by the previously described program would require approximately 25 milliseconds on the DDP-24. Including the fixed point operations required by this program, a safe estimate of the running time per decision interval is in the neighborhood of 30 milliseconds. A simulation run of 40 decision intervals would take on the order of 1.2 seconds of computer time. However, of more importance is the fact that real time control is approaching the realm of feasibility with this program execution time.

If the computations for the unit step response block are included the program execution time would be increased by 5 to 15 per cent. Therefore, the DDP-24 computation time per decision interval would be no greater than 35 milliseconds. The addition of the judgement functions, g (Δ m) and f(Δ m), would most likely increase the computer time by somewhere between 20 and 40 per cent. Also, if the program is expanded for plants of tenth or lower order the execution time would be increased by approximately 150 to 200 per cent.

Several methods could be employed to shorten the computer time per interval. Instead of calculation of the unit step response, $a_i(o)]$, every interval it could be computed only every "n" intervals. Also, another way would be to control to lower than actual order; i.e. instead of estimating the function and all of its derivitives, estimate only the function and its first few derivatives. The most obvious way to decrease computer time would be to program in fixed point arithmetic, in which case the computer time would be less than 5 milliseconds. However, this does create scaling problems if a large class and/or number of plants are to be controlled.

V. MASTER GLOSSARY

The following master glossary has been prepared containing the nomenclature of this and all prior progress reports. The previous quarterly reports contain partial glossaries; and a conscious attempt had been made towards standardization of notation throughout the various studies. However, the diversity of authorship of the various studies, their extension in time, and an observed prolification of symbols suggested a systematization.

In preparing the glossary very few redundant symbols were discovered. In the few cases where two symbols have identical meaning, the identity equation has been written under each entry.

A more common problem is the use of a given symbol in radically different meanings. In such cases, the alternate definitions both appear under the entry, together with reference to those portions of the progress report to which each applies.

The hierarchy of symbolism of this glossary is as follows:

(1) English alphabetic symbols - Greek alphabetic symbols - Notational conventions

(2) Lower case - Upper case

(3) [illegible]

(4) General arguments - Specific arguments

An entry of the glossary contains:

(1) The symbol

(2) A verbal description or definition

(3) A defining equation or equations (where applicable)

(4) A reference to the progress report or study, where the symbol is first defined and used.

References to the reports are given numerical designations (1), (2), (3), (4), respectively designating the four quarterly progress reports (Emerson Report Numbers 1544 - 1, 2, 3, 4). Additionally origin in Digital Simulation, Hybrid Simulation, or Stability Studies, is indicated by those phrases.

This glossary is believed to be exhaustive, with the exception that a few symbols used in digital simulation studies were deliberately omitted as being peculiar to the computer programming.

SYMBOL	INFORMATION	REFERENCE
$a(P_k)$	Numerator of plant transfer function evaluated at k'th pole.	Digital Simulation (1)
a_i	Elements of $\bar{a}$	(1)
$a_i(0)$	Estimate of i'th component of unit step response vector $\bar{a}(0)$ obtained by averaging over past values. Form of averaging finally used: $$a_i(0) = \frac{\sum_{k=2}^{N} \left\| C^{(i)}(-kT) - C_f^{(i)}(-kT) \right\|}{\sum_{k=2}^{N} \left\| m(-(k+1)T) \right\|}$$	(1)
	Also equivalent form used in simulation $$a_i(0) = \frac{F_i(0)}{G_1(0)}$$	Digital Simulation (1)
$a_i(-T)$	Estimated unit step response vector one interval in past obtained by averaging over past values. Has elements: $$a_i(-T) \left(\text{cf. } a_i(0)\right)$$	Hybrid Simulation (3)
$a_k^r(t)$	Cumulative sum of integrated kernels characterized by having an index (and control action u_k) repeated r times. $$a_k^r(t) = \frac{1}{U_k^{\,r}} \sum_{\langle k \rangle} A_{\langle k \rangle} M_{\langle k \rangle} U_{\langle k \rangle}$$	(4)
$\bar{a}$	Constant value of $\bar{a}(kt)$ for time invariant plants	(2)
$\underline{a}$	Sensitivity vector, ratio of state vector change to previous control force. $$\underline{a} = \left[\frac{\underline{x}(nT) - \underline{A}\,\underline{x}((n-1)T)}{u_{n-1}} \right]$$	(4)
$\bar{a}(kT)$	Estimated change in the weighted response state due to the force m (kt) $$\bar{a}(kT) = \frac{\bar{x}_m\left[(k+1)T\right]}{m(kT)}$$	(2)
$\bar{a}(k)$	$\bar{a}(k) \equiv \bar{a}(kT)$	(2)
$\bar{a}(0)$	Estimated value of the change in the weighted response due to unit control force. Obtained by averaging past values of: $$\bar{a}(-iT) = \frac{\bar{x}_m\left[-(i-1)T\right]}{m(-iT)}$$ $$\bar{a}(0) = \bar{A}\left[\bar{a}(-2T), \ldots \bar{a}(-NT)\right]$$	(1)
$\underline{a(i)}$	Exact unit step response vector at t = iT	Stability Study (3)

SYMBOL	INFORMATION	REFERENCE

$\bar{a}\ (-2T)$ Estimated ratio of change in forced response state to applied force for most recent interval:

$$\bar{a}\ (-2T) = \frac{\bar{x}_m\ (-T)}{m\ (-2T)} = \frac{\bar{x}\ (-T) - \bar{x}\ (-2T) - \bar{d}(-2T)}{m\ (-2T)} \qquad (1)$$

$A\ (k)$ Weighted state transition matrix relating free response at end of decision interval, to state at start of interval.

$$\bar{x}_f\ \left[(k+1)\right] = \left[A\ (k)\right]\ \bar{x}\ (k) \qquad (2)$$

$\bar{A}_{k_1 \ldots k_j}\ (t)$ Integrated kernels of order j of Volterra functional polynomial fit

$$\bar{A}_{k_1 \ldots k_j} = \begin{cases} 0 & t < \chi T \\ \int_{k_1 T}^{K} \cdots \int_{k_j t}^{K} h_j\ (t, \tau_1, \ldots \tau_j)\ d\tau_1 \cdots\ d\tau_j & \chi T \leq t \leq (\chi+1)\ T \\ \int_{k_1 T}^{(k_1+1)T} \cdots \int^{(k_j+1)T} h_j (t, \tau_1, \ldots \tau_j)\ d\tau_1 \cdots d\tau_j & (\chi+1)\ T \leq t \end{cases} \qquad (4)$$

$A_{<k>}$ Integrated kernels of Volterra functional polynomial fit.

$A_{<k>} = A_k^{r_1}\ (k-1)^{r_2} \cdots (k-R)^R$ (t) denotes one of the $A_{k_1 \ldots k_j}$ with k repeated $r = r_1$ times, $(k-1)\ r_2$ times, etc. (4)

$A_{<k>\ p\pm}$ Coefficients of Taylor expansion of integrated kernels $A_k\ (t)$

$$\bar{A}_k = \begin{cases} 0 & t < kT \\ \sum_{p=o}^{p} A_{<k>p-} (t-kT)^p & kT \leq t \leq (k+1)\ T \\ \sum_{p=o}^{P} A_{<k>p+} (t-kT)^p & (k+1)\ T < t \end{cases} \qquad (4)$$

$\bar{A}_{<n>\ p\pm}$ Coefficients in the expansion of the Taylor coefficients $A_{<k>\ p\pm}$ into:

$$A_{<k>\ p\pm} = \sum_{s=o}^{S} A_{<n>\ p\ s\pm}\ (-hT)^s \qquad (4)$$

(i)
A_{nj} i'th state component of integrated Volterra kernel evaluated at $t = (n+1)\ T$

$$\overset{(i)}{A}_{nj} = \frac{d}{dt^i} \int_{nT}^{t} \cdots \int_{nT}^{t} h_j\ (t, \tau_1, \ldots \tau_j)\ d\tau_1, \ldots\ d\tau_j \Big|_{t = (n+1)\ T}$$

(4)

SYMBOL	INFORMATION	REFERENCE
$\underline{A}$	Taylor series prediction matrix $A_{ij} = \begin{cases} T^{j-i} / (j-i)! & i < j \\ 1 & i = j \\ 0 & i > j \end{cases}$ $1 \geq i \geq P+1$ $1 \geq j \geq N$ $P+1 > N$	(4)
$[A]$	Exact state - transition matrix	Stability Study (3)
$\underline{A}_k(t)$	Matrix of integrated Volterra kernels $\underline{A}_k(t) = a_k^{(i)r}(t) \Big]_{N \times R}$	(4)
$\underline{A}_n(t)$	Matrix of integrated kernels evaluated at $t = (n+1)T \quad \underline{A}_n(t) = \left[{}^{(i)}_{A_n j} \right]$	Stability Study (4)
$[AA_{ij}]$	Transformation matrix for estimate of free response at the end of present interval $C_f{}^i(0)] = [AA_{ij}] C^{(i)} [-T]$ $AA_{ii} = T^{j-i}/(j-i)!$ for $i \leq j$ For Taylor series prediction $AA_{ij} = BB_{ij}$ for exact prediction	Digital Simulation (1)
$b'(p_k)$	Derivative of denominator of plant transfer function with respect to LaPlace operator evaluated at the k'th pole	Digital Simulation (1)
$\underline{b}'(u_{n-1})$	Combinational vector $u_n = b'(u_{n-1}) \underline{x}(nT)$ $u_n = \dfrac{\sum_{j=1}^{J} \left(u_{n-1}^{j-1} \underline{A}_n j \right) \underline{KA}}{\sum_{j-1}^{J} \sum_{i-1}^{J} u_{n-1}^{j+i-2} \underline{A}_n i \underline{K} \underline{A}_n j}$	Stability Study (4)
$\hat{b}(i)$	Represents best available estimate of $\underline{b(i)}$.	Stability Study (3)
$\underline{b(i)}$	Unit step response Vector as defined by $\underline{a(i)} = \underline{b(i)} [h]$	Stability Study (3)
$\underline{B}$	Computed force vector extrapolated one prediction interval into the future $\underline{B} = \underline{c} [AA_{ij}]$	Hybrid Simulation (3)
$[B]$	Exact state-transition matrix as defined by $[A] = [h] [B] [h]^{-1}$	Stability Study (3)

SYMBOL	INFORMATION	REFERENCE
$\hat{[B]}$	Best available estimate of $[B]$.	Stability Study (3)
$[BB_{ij}]$	Exact plant response matrix for linear systems	
	$BB_{ij} = \sum_{k=1}^{J} \sum_{\ell=j+1}^{J} \left[b_i \, p_k^{\ell-j-1+i} \Big/ b'(p_k) \right] \exp(p_k T)$	Digital Simulation (1)
$c(t)$	Response variable also $C(t)$	(1)
$c_{ei}(0)$	Estimated value of i'th component of initial state response applied one decision interval, T, earlier	Digital Simulation (1)
$c_{ei}(T)$	Estimated value of i'th state component of initial state response evaluated for time one interval from present.	Digital Simulation (1)
$c_{mi}(0)$	Actual value of i'th state variable in response to force applied T earlier	Digital Simulation (1)
$c_{oi}(0)$	Actual value of i'th state variable in response to initial conditions applied one decision interval, T, earlier	Digital Simulation (1)
$c^{(i)}(0)$	Total response over one interval	
	$c^{(i)}(0) = c_{oi} + c_{mi}$	Digital Simulation (1)
$\bar{c}$	Response state vector	(1)
$\bar{c}(0)$	Estimated present state of system	(1)
$\underline{c}$	Computed force vector at present time	
	$\underline{c} = \dfrac{\underline{a_i}(0) \; [K]}{\underline{a_i}(0) \; [K] \; a_i(0)]}$	Hybrid Simulation (3)
$\bar{c}_f(T)$	Estimated free response state at the next future decision interval	
	$C_f^{(i)}(T)$ = i'th component of $\bar{C}_f(T)$	Hybrid Simulation (2)
$c^{(i)}(-T)]$	Response state vector evaluated at one interval in past	Hybrid Simulation (3)
$c_f^{(i)}(0)]$	Estimate of free response vector at end of present interval	
	$c_f^{(i)}(0)] = [AA_{ij}] \; c^{(i)}(-T)]$	Hybrid Simulation (3)
$\underline{c}\ ((n+1)\,T)$	Desired value of vector $\underline{x}$ at $t = (n+1)\,T$ of dimension N	(4)
$C(t)$	Response variable $C(t) \equiv c(t)$	(1)
C_i	Value of i'th response state variable	(2)

SYMBOL	INFORMATION	REFERENCE
CC_j	j'th component of exact plant forced response to unit control force $CC_j = \sum_{k=1}^{J} \left[a(p_k) / b'(p_k)\right] p_k^{j-1} \exp(p_k T)$ $J-1 \geq J \geq 1$ $CC_o = a_o / b_o + \sum_{k=1}^{J} a(p_k) / p_k b'(p_k) \exp(p_k T)$	Digital Simulation (1)
$C^{(i)}(-T)]$	Response state vector evaluated at one decision interval in the past	Hybrid Simulation (2)
$\bar{d}(0)$	Estimated change in weighted state vector due to free response evaluated at present decision interval	
$\bar{d}(k)$	$\bar{d}(k) = \bar{d}(kT)$	(1)
$\bar{d}(kT)$	Estimated change in response state vector over one interval due to the initial state $\bar{x}(kT)$	(2)
$\underline{D}$	Combinational matrix $\underline{D} = \left[\underline{I} + \frac{\underline{a}\,\underline{a}'\,\underline{K}}{\underline{a}'\,\underline{K}\,\underline{a}}\right] \underline{F}$ Stability requires that the eigenvalues of $D < 1$	Stability Study (4)
$[D]$	Simplified way of writting resultant matrix $[D] = \frac{+ b(i)] \quad \underline{\hat{b}(i)} \quad [H] \quad [\hat{B}]}{\underline{\hat{b}(i)} \quad [H] \quad \hat{b}(i)]}$	Stability Study (3)
$[DD]$	Transformation matrix for estimate of free response at end of next interval $[\quad] \quad - \quad [\quad] \quad [\quad]$	Hybrid Simulation (2)
e	Index of expansion $\underline{x}(t) = \sum_{e=0}^{P} g_{ke} t^e$	(4)
$e_i(0)$	Error (actual, measured, or computed) between the i'th state variable of the desired trajectory and the i'th state variable of the actual trajectory	Digital Simulation (1)
$e_{1i}(T)$	Predicted error between i'th reference state variable and i'th initial state vector, evaluated for one decision interval in the future $e_{1i}(T) = r_p^{(i)}(T) - C_f^{(i)}(T)$	Digital Simulation (1)
$\bar{e}$	Error state vector $\bar{e} = \bar{r} - \bar{c}$	(1)
$\bar{e}_p(T)$	Predicted error state vector at the next interval	(1)
E_n	Change of the error norm between $t = nT$ and $t = (n+1)T$ $E_n = \|\|\underline{x}((n+1)T)\|\|^2_{\underline{H}} - \|\|\underline{x}(nT)\|\|^2_{\underline{H}}$	(4)

SYMBOL	INFORMATION	REFERENCE
$\bar{E}(T)$	Weighted predicted error state vector at next interval $\bar{E}(T) = \bar{Y}_p(T) - \bar{x}(0) - -\bar{d}(0) - \bar{a}(0)\ m(0)$	(1)
$[E] = [B] - [D]$	Matrix simplification used in equations	Stability Study (3)
$\|\bar{E}(k+1)\|$	Norm of error state vector evaluated at $t = (k+1)T$ $\|\bar{E}(k+1)\|^2 = \bar{x}((k+1)T) \cdot \bar{x}((k+1)T)$	(2)
f	A design function of the argument $\left[m_a(0) - m_a(-T) \right]$ to prevent large corrections which might be caused by measured errors in one interval	(1)
$f_i(c,\dot{c},\ldots c^{(J)},m,t)$	The function coefficient of the derivative $c^{(i)}$ in the plant differential equation	(1)
$\underline{F}$	Exact state transition matrix for stationary system $\underline{F} = \underline{F}\ (nt,(n+1)T)$ for stationary system	Stability Study (4)
$\underline{F}(nT,(n+1)T)$	Exact state transition matrix between states defined by $t = nT$ and $t = (n+1)T$	Stability Study (4)
$F_i(0)$	Cumulative vector sum of absolute values of $\Delta c_i(0)$ for previous decision times $F_i(0)] = [\Delta\, c^{(i)}(-T)[] + F_i(-T)]$	Digital Simulation (1)
g	Design function of the argument $\left[\frac{m_a(0) - m(T)}{m(-T) - m(-2T)}\right]$ it is intended to emphasize changes following a well behaved pattern and de-emphasize erratic changes. It is symmetric around the argument value 1, where it is maximum.	(1)
$g_i(c_o,c_1,\ldots c_{J-1}, m, t)$	The function defining the state variable c_i in the differential equation set $\dot{c}_i = g_i\ (c_o, c_1, \ldots c_{J-1}, m, t)$	(1)
g_{ke}	Vector of Taylor coefficients in expansion $\underline{x}\ (t) = \sum_{e=o}^{P} g_{ke}\ t^{e} \qquad kt < t < (k+1)t$	(4)
G	LaPlacian operator form of plant transfer function $G = \frac{C(s)}{E(s)}$ for linear systems	(1)
$G_1(o)$	Cumulative summation of absolute values of control force applied at past decision times $G_1(0) = \|m(-2T)\| + G_1(-T)$	Digital Simulation (1)
$\underline{G}(i)$ or $[G(i)]$	Combination of other matrices as defined by: $\underline{G}(i) = \underline{E}'(i)\ \underline{K}\ \underline{E}\ (i)\ -\underline{K}$	Stability Study (3)

SYMBOL	INFORMATION	REFERENCE
h	Error state weighting factor	
	$x_i = h^i \; WN_i \; c^{(i)}(t)$	Digital Simulation (1)
	Index in expansion	
	$A_{\langle k\rangle\, p\pm} = \sum_{s=0}^{S} A_{\langle n\rangle\, ps\pm} \; (-hT) \qquad h=n-k$	(4)
$h_j(t, \tau_1, \ldots \tau_j)$	Kernel of order j of Volterra functional polynomial fit	(4)
$[h]$	Weighting factor as defined by:	
	$[h] = [h^j \delta_{jk}]$	Stability Study (3)
H	Error state weighting factor. The transformation matrix $\bar{w}$ is a diagonal matrix with elements	
	$w_{ii} = W_i = H^i \qquad h < 1 \qquad \bar{E}(T) = \bar{W} \cdot \bar{e}_p(T)$	(1)
$[H]$	Weighting function matrix	
	$[H] = [h^{2j} \delta_{jk}] = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & h^2 & \vdots \\ 0 & \cdots & h^{2n} \end{bmatrix}$	Stability Study (3)
I	Unit matrix	(2)
$I_{\{k_i k_j\}\, z}$	Integrated squares of residuals in mean square determination of coefficients	
	A and y	
	$I_{\{k_i k_j\}\, z} = \int_{k_i T}^{k_j T} \left[\hat{\xi}_z(t) - \left(\sum_{e=z}^{P} \frac{e!}{(e-z)!} \, g_{ke} \, t^{e-z} \right) \right]^2 dt$	(4)
J	Order of plant differential equation	(1)
	Maximum order of Volterra kernel $h_J(t, \tau_1, \ldots \tau_J)$	(3)
$\langle k\rangle$	Index set	
	$\langle k\rangle = \langle k_1, k_2 \ldots k_j \rangle$ and $k_{i-h} \geq 0$	(4)
$\langle k-h\rangle$	Index set	
	$\langle k-h\rangle = \langle k_1-h, k_2-h, \ldots k_j-h\rangle$	(4)
K_1	Norm of weighted estimate of unit step response vector $a_i(0)$	
	$K_1 = \underbrace{w_i a_i(0) \quad w_i a_i(0)}$	Hybrid Simulation (2)

SYMBOL	INFORMATION	REFERENCE
K_2	Product of the predicted weighted error vector by weighted estimate of unit step response vector	
	$K_2 = \underline{w_i a_i}(0) \quad w_i e_{1i}(T)\Big]$	Hybrid Simulation (2)
$[K]$	Weighting matrix	
	$K_{ij} = \begin{cases} (Th)^{2i} & i = j \\ 0 & i \neq j \end{cases}$	Hybrid Simulation (3)
$[K]$ or $\underline{K}$	Positive definite (symmetric) matrix	Stability Study (3)
m	Number of zeroes in plant transfer function	Digital Simulation (1)
m (0)	Actual force applied over the next interval	
	$m_o = \begin{cases} m_o & m'(0) > m_o \\ m'(0) & \text{if } -m_o \leq m'(0) \leq m_o \\ -m_o & m'(0) < -m_o \end{cases}$	(1)
m(t)	Control force or manipulated variable	(1)
m_k	Stationary uncorrelated noise perturbing the measurement or computation of the control force u_k	(4)
m_o	Saturation force value of the controller	
	$m_o = U$	(1)
m(k)	$m(k) = m(kt)$	(2)
m (kT)	Control force applied over k'th interval	
	$kT \leq t < (k+1)T$	(1)
$m_a(0)$	Computed or [illegible] to be applied over next interval	
	$m_a(0) = -K_2/K_1$	Digital Simulation (1)
	$m_a(0) = \underline{c}\, r_p^{(i)}(T)\Big] -$	
	$\underline{B}\, c^{(i)}(0)$	Hybrid Simulation (3)
$m_2(0)$	Computed force over the next interval modified by the f and g functions	
	$m_2(0) = m_a(0) + f(\Delta m)\, g\, (\Delta m)$	Digital Simulation (1)
m'(0)	Modification of the computed force	(1)
M <k>	Multiplicity of occurance of term $A_{<k>}$	
	$M_{<k>} = \sum_{r_2=0}^{R-r_1} \sum_{r_3=0}^{R-r_1-r_2} \cdots \sum_{r_R=0}^{R-r_1 \cdots -r_R} \frac{\left(\sum_{i=1}^{R} r_i\right)!}{\prod_{i=1}^{R} r_i\,!}$	(4)

SYMBOL	INFORMATION	REFERENCE
$\bar{m}$	Manipulated variable state vector	(1) (2) (3)
$\bar{m}(kt)$	Manipulated variable state vector evaluated at decision interval kT	
	$\bar{m}\ (kT) = \delta(i = 2, m\ (kT)$	(2)
Δm	Increment in applied force $\Delta m = m_a\ (0) - m\ (-T)$	Digital Simulation (1)
M	Combination matrix yielding change in error norm at t = (n+1)T from state vectors at (nT)	
	$E_{n+1} = -\underline{x}'\ (nT)\ \underline{M}\ \underline{x}\ (nT)$	
	$\underline{M} = \underline{H}\ \underline{-F}'\ (nT,\ (n+1)T)$	
	$\left[\underline{I} + \frac{\underline{K}'\ \underline{a}\ ((n-1)T)\ \underline{a}'\ ((n-1)T)}{\underline{a}'\ ((n-1)T)\ \underline{K}\ \underline{a}\ ((n-1)T)}\right] \underline{H}$	
	$\left[\underline{I} + \frac{\underline{a}\ ((n-1)T)\ \underline{a}'\ ((n-1)T)\ \underline{K}}{\underline{a}'\ ((n-1)T)\ \underline{K}\ \underline{a}\ ((n-1)T)}\right]$	
	$\underline{F}\ (nT,\ (n+1)\ T)$	Stability Study (4)
n	Number of samples since beginning of run	Digital Simulation (1)
$\underline{n}\ (t)$	State vector of stationary uncorrelated noise perturbing measurement of exact state vector $\underline{x}\ (t)$	(4)
N	Dimension of vectors $\underline{x}$ and $\underline{c}$	(4)
p_i	Poles of plant transfer function	Digital Simulation (1)
	$A_k(t)$, and x (t)	(4)
r	Index variable designating number of times a specific index k is repeated in the control sequence u (t)	(4)
r_i	Index variable. When the indices k, k-1, ...0 are utilized, the number of repetitions of index k are designated by $r = r_1$; those of index (k-1) are designated by r_2; etc.	(4)
$\bar{r}$	Reference (input) state vector	
	$\bar{r} = \begin{bmatrix} r \\ \dot{r} \\ \vdots \\ r^{(n)} \end{bmatrix}$	(1)
$\bar{r}(0)$	Estimated present value of input or reference state vector	(1)
$\bar{r}_p(T)$	Estimated predicted value of the desired state one decision interval into the future	(1)
$r^{(i)}(-t)]$	Reference state vector evaluated at one decision interval in the past	Hybrid Simulation (2)

SYMBOL	INFORMATION	REFERENCE
R	Maximum value of number of repetitions of a single control variable u_k in Volterra series representation usually $R = J$	(4)
s	LaPlace variable	(1) (2) (3) (4)
	Also index of expansion of $A_{<k>\ p\pm}$ in $A_{<k>\ p\pm} = A_{<n-h>\ p\pm} = \sum_{s=0}^{S} A_{<n>\ ps\pm} (-hT)^s$	(4)
S	Maximum value of index s in expansion $A_{<k>\ p\pm} = A_{<n-h>\ p\pm} = \sum_{s=0}^{S} A_{<n>\ ps\pm} (-hT)^s$	
	S = 1 for most plants studied	(4)
t	Time	(1) (2) (3) (4)
	When shifting time base is used; preceding decision point is t = -T, current decision point is t = 0, and next decision point is t = T	(1)
Δt	Sampling interval of data inputted to control computer	Digital Simulation (1)
T	Decision interval. This is a fixed interval of time which quantizes the control actions. The control force is constant over any given decision interval.	(1)
u (t)	Control variable time function	
	$u(t) = u_k \quad kT \leq t < (k+1)T$ and $\lvert u_k \rvert \leq U$	(4)
u_i	Error state element	
	$u_i = T^i\ c(i)$ for reference input $\bar{r} = 0$	(2)
u_k	Constant value of control variable in interval $kT \leq t < (k+1)T$; Also constrained by: $\lvert u_k \rvert \leq U$ Also identified with previously defined variables $u_k \equiv m\ (kT) \equiv m\ (k)$	
	Alternately: Coefficients of expansion of quadratic approximation for ideal control force v_o in powers of $v_o = u_o + u_1\delta + u_2\delta^2 + \ldots$	
	$u_o = -1/\alpha$	
	$u_k = -\left[\frac{1}{\alpha^{2k+1}}\right] q_k$ where	
	$q_k = \sum_{i=0}^{k-1} q_i\ q_{k-i-1}$ for $k > 0$	
	$q_o = 1$	(4)

SYMBOL	INFORMATION	REFERENCE
u_k^r	Constant value of control force u_k iterated r times in a control sequence u(t)	
	$u_k^r \equiv (u_k)^r$	(4)
$\underline{u}_k$	Control variable vector in interval $kT \leq t < (k+1)T$	
	$\underline{u}_k = u_k^r \Big]_{Rx1}$	(4)
U	Saturation limit of control force	
	$\lvert u_k \rvert \leq U$	
	$U \equiv m_o$	(4)
U_i	Normalized value of i'th response state variable	
	$U_i = T^i C_i$	(2)
$U_{<k>}$	Repetitive control variable product function	
	$U_{<k>} = u_k^{r_1} u_{k-1}^{r_2} \cdots u_{k-R}^{r_R}$	(4)
v_k	Coefficients of expansion of cubic approximation for ideal control force u in powers of Δ	
	$u = v_o + v_1 \Delta + v_2 \Delta^2 + \ldots$	
	v_o cf . u_k	
	$v_1 = \dfrac{-v_o^3}{\alpha + 2\delta v_o}$	
	$v_2 = -\dfrac{3v_o^2 v_1 + v_1^2}{\alpha + 2\delta v_o}$	
	$v_k = -\dfrac{\sum_{i=1}^{k-1} \quad i\ k-j \quad \sum_{i=0}^{k-1} \sum_{j=0}^{k-i-j} \quad i\ j\ k-i-j-1}{\alpha + 2\delta v_o}$	(4)
v (iT)	Liapunov function assumed in quadratic form	
	$v(iT) = \underbrace{Z\ (iT)}\ [K]\ Z\ (it)]$	Stability Study (3)
ΔV	Free response component of norm change E_n	
	$E_n = \Delta V \left(\underline{x}\ (nT) \right) + \Delta W \left(\underline{x}(nT),\ u_{n-1} \right)$	Stability Study (4)
w_{ni}	Weight for i'th state variable	
	$w_{ni} \equiv W_i = T^i h^i$	Digital Simulation (1)
W_i	Diagonal elements of transformation matrix W Alternate choices investigated include	(1)
	$W_i = T^i h^i / i!$	
	$T^i h^i\ (J-i)!/J!$	
	$T^i h^i$	(2)
	H^i	

SYMBOL	INFORMATION	REFERENCE
$\bar{W}$	Diagonal transformation matrix with elements $w_{ii} = W_i$ $\bar{E}\ (T) = \bar{W}\ e_p(T)$	(1)
WN_i	Elements of norm weighting transformation matrix $x_i = WN_i\ h^{i}\ c^{(i)}(t)$ Alternate choices investigated include: $WN_i = T^{i}/i\ !$ $= T^{i}\ (J-i)!/J!$ $= T^{i}$	(2)
ΔW	Forced response component of norm change from $t = nT$ to $t = (n+1)T$. $E_n = \Delta V(\underline{x}\ (nT)) + \Delta W\ \ (\underline{x}(nT),\ u_{n-1})$	Stability Study (4)
$x\ (t)$	Total control system output (associated with Volterra Series description)	(4)
$x_i(0)$	Estimated weighted response i'th state component $x_i(0) = w_{ni}\ e_i\ (0)$	Digital Simulation (1)
$\underline{x}$	Output state vector $\underline{x} = x^{(i)}\ \big]\ N \times 1$ Also $\underline{x} \equiv \underline{x}(t)$	(4)
$\underline{x}\ (t)$	Output state vector $\underline{x}\ (t) \equiv \underline{x}$	(4)
$\bar{x}(0)$	Estimated initial weighted response state vector	(1)
$\bar{x}(k)$	$\bar{x}(k) \equiv \bar{x}(kT)$	(2)
$\bar{x}(kT)$	Estimated weighted response state vector at decision time $t = kT$	(2)
$\bar{x}_f(T)$	Weighted estimated free response state vector	(1)
$\bar{x}_m$	State vector of forced response only	(1)
$\bar{x}_m(-T)$	Change in forced response state over interval $-2T \leq t \leq -T$	(1)
$\underline{x}_n(t)$	Partial output state vector $\underline{x}_n(t) = \underline{y}(t) + \sum_{k=0}^{n-1} \underline{A}_k(t)\ \underline{u}_k$	(4)
$\left\|\left\| \underline{x}(nT) \right\|\right\|_H^2$	Weighted norm of state vector $\underline{x}\ (uT)$ $\left\|\left\| \underline{x}(nT) \right\|\right\|_H^2 = \left[\underline{x}'(nT)\right]\ \underline{H}\ \left[\underline{x}(nT)\right]$	(4)
X_i	Value of weighted normalized i'th state variable $X_i = (T^i h^i\ \ (J-i)!\ /\ J!)\ C_i$	Parameter Studies (2)
$y(t)$	Free response in absence of any control input	(4)

SYMBOL	INFORMATION	REFERENCE
y_p	p'th Taylor coefficient in expansion of y(t)	
	$y(t) = \sum_{p=0}^{P} y_p t^p$	(4)
$\underline{y}$	Free response state vector $y = y^{(i)}]$ N x 1	
	Also $\underline{y} \equiv \underline{y}(t)$	(4)
$\underline{y}(t)$	Free response state vector $\underline{y}(t) \equiv \underline{y}$	(4)
$Y(0)$	Estimated present value of weighted input or reference state vector	(1)
$\bar{Y}_p(T)$	Estimated weighted desired state vector one decision interval into the future	(1)
z_i	Zeroes of plant transfer function	Digital Simulation (1)
$Z(i)]$	Response state vector at decision interval $t = iT$	
	$Z(i)] = [WN_j \quad \delta_{jk}] \; c(i)]$	Stability Study (3)
α	Coefficient of linear term in second and third order approximation of roots of control equation for u_n	(4)
	Second order approximation	
	$1 + \alpha v_o + \delta v_o^2 = 0$	
	Third order approximation	
	$1 + \alpha u + \delta u^2 + \Delta u^3 = 0$	(4)
δ	Coefficient of quadratic term in second and third order approximations of roots of control equation for u_n	
	cf. α $\quad \delta < \alpha$	(4)
Δ	Coefficient of cubic term in third order approximation of roots of control equation for u_n	
	cf. α $\quad \Delta < \alpha$	(4)
η (eta)	Number of $<n>$ sets which are considered significant	(4)
K (Kappa)	Variable upper limit of integration in Volterra functional series representation	
	$K = \begin{cases} (k_i+1)\,T & \text{for } k_i < X \\ t & \text{for } k_i = X \end{cases}$	(4)
λ_z (lamda)	Weighting factors in mean square determination of coefficients $A_{\langle n \rangle\, ps\pm}$ and y_p	(4)
ν_k (nu)	Measured on computed value of control force u_k as contaminated by stationary uncorrelated noise m_k	(4)
$\xi(t)$ (xi)	Measured value of the state vector $\underline{x}(t)$ as contaminated by stationary, uncorrelated noise $\underline{n}(t)$	
	$\xi(t) = \underline{x}(t) + \underline{u}(t)$	(4)

SYMBOL	INFORMATION	REFERENCE
τ_i (tau)	Dummy time variables in Volterra series	(4)
χ(chi)	Maximum index of given control sequence	
	$\chi = \max \{k_1, k_2, \ldots k_j\}$	(4)

Derivative Notation

$\vartheta^{(i)}$	i'th time derivative of a variable ϑ. If $\bar{\vartheta}$ is a state vector this is i'th component.	(1)

Difference Notation

$\Delta\zeta$	Change in a variable (ζ) over one previous decision interval.	
	Example: $\Delta m(0) = m(0) - m(-T)$	(1)

Time Conventions:

t denotes a continuous time variable. For a given control action, the origin of t is nT decision intervals in the past. Thus t has a floating origin. The designations t = kT denote the (n+1) decision intervals in the past. The arguments (t) and (kT) are based on this time convention.

A different time convention with floating origin at time of present control action is also used, particularly in the simulation studies. In this convention the arguments (-T) (0), (T), respectively designate times: one decision interval in the past, the present decision epoch, and one decision interval in the future.

Vector Notation: $\bar{v}$ or $\underline{v}$ designates a vector which may be in row or columnar form.

$\bar{v}'$, $\underline{v}'$	designates the transpose of $\bar{v}$ or $\underline{v}$	
$v(i)\rfloor$	explicitly designates the columnar form	
$\underbrace{v^i}$	explicitly designates the vow form	
$\|\bar{v}\|$	norm of a vector $\bar{v}$. Usually $\|v\| = \bar{v}.\bar{v}$.	(1)

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